Nodal solutions for an elliptic problem involving large nonlinearities *

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Abstract

Let us consider the problem

$$\begin{cases} -\Delta u + a(|x|)u = |u|^{p-1}u & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{0.1}$$

where B_1 is the unit ball in $\mathbb{R}^N$, $N \geq 3$ and $a(|x|) \geq 0$ is a smooth radial function. Under some suitable assumptions on the regular part of the Green function of the operator $-u'' - \frac{N-1}{r}u + a(r)u$ we prove the existence of a changing sign solution to (0.1) for p large enough.

Keywords: Supercritical problems, Green's function, radial solutions.

1 Introduction

In this paper we prove the existence of changing sign solutions to the problem

$$\begin{cases} -\Delta u + a(|x|)u = |u|^{p-1}u & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{1.1}$$

where B_1 is the unit ball of $\mathbb{R}^N$, $N \geq 3$, $p > 1$, $a(|x|) \geq 0$ is a smooth function.

It is known that the existence of solutions to (1.1) strongly depends on the value of p and by the shape of the function a.

A classical result states that if $1 < p < \frac{N+2}{N-2}$ (subcritical case) there exist infinitely many solutions to (1.1) (the same holds if we replace B_1 by a general domain Ω in $\mathbb{R}^N$.)

When $p = \frac{N+2}{N-2}$ (critical case) the problem looks like different. Indeed, using the Pohozaev identity ([P]) it is possible to show that (1.1) has no solution if $a(|x|) \equiv 0$. A fundamental existence result of positive solutions in the critical case was established in the pioneering paper of Brezis and Nirenberg ([BN]).

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Some existence result for changing sign solutions when $a(|x|)=\lambda<0$ and $p=\frac{N+2}{N-2}$ were proved in [CFP], [CFS], [CSS], .

If $p>\frac{N+2}{N-2}$ (supercritical case) the problem becomes more difficult to the lack of embedding of the space $H_0^1(B_1)$ in $L^{p+1}(B_1)$. Note that we have some existence result for positive solution just for p slightly supercritical ([DDM]) or in suitable domains (see [DW],[DW1] [P1], [P2], [P3]).

To our knowledge there is no result for changing sign solution in the unit ball for the supercritical case. The aim of this paper is to give a contribution in this direction, following the research started in [G1] and [G2]

Let us denote by $G(r,s)$ the Green function of the operator $-u''-\frac{N-1}{r}u'+a(r)u$ in $(0,1)$ with $u(1)=0$ and with $H(r,s)$ its regular part (see Section 2 for more details).

In [G2] it was studied the existence of a radial positive solution to (1.1) when the exponent p is large and it was proved that if there exists a nondegenerate critical point $\bar r$ of the function

$$F(r)=\frac{H(r,r)}{r^{N-1}}$$

then there exists a radial positive solution u_p to (1.1) which satisfies

$$u_p(r)\to\frac{G(r,\bar r)}{H(\bar r,\bar r)}\quad\text{uniformly in }(0,1).\tag{1.2}$$

The main tool of the previous result is the classical contraction mapping theorem. This approach was used by many authors in similar problems and it involves hard and tedious computations.

In this paper we will use the result in [G2] and we give some sufficient conditions which ensure the existence of a changing sign solution. We stress that we will not use any finite dimensional reduction.

Indeed we provide an alternative proof which is, in the opinion of the author, much simpler.

Before of stating our main result we describe the main ideas of the paper.

Let $z\in(0,1)$ and $u_{1,p}$, $u_{2,p}$ solutions of

$$\begin{cases}-u''-\frac{N-1}{r}u'+a(r)u=u^p & \text{in }(0,z),\\ u>0 & \text{in }(0,z)\\ u'(0)=u(z)=0,\end{cases}\tag{1.3}$$

and

$$\begin{cases}-u''-\frac{N-1}{r}u'+a(r)u=u^p & \text{in }(z,1),\\ u>0 & \text{in }(z,1)\\ u(z)=u(1)=0,\end{cases}\tag{1.4}$$

respectively. For the moment we avoid to write the conditions which ensure the existence of $u_{1,p}$. Then we set

$$u_p(r)=\begin{cases}u_{1,p}(r) & \text{if }r\in(0,z),\\ -u_{2,p}(r) & \text{if }r\in(z,1).\end{cases}\tag{1.5}$$

Of course we have that u_p is a solution to

$$\begin{cases} -u'' - \frac{N-1}{r}u' + a(r)u = u^p & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.6}$$

if and only if the equality

$$u'_{1,p}(z) = u'_{2,p}(z) \tag{1.7}$$

holds for some $z \in (0,1)$. In order to find such a z we pass to the limit in (1.7) and, using the asymptotic behavior of $u_{1,p}$ and $u_{2,p}$ (Section 2 and 3) we have

$$\frac{\overline{G}_r(z, r_1)}{\overline{H}(r_1, r_1)} = \frac{\underline{G}_r(z, r_2)}{\underline{H}(r_2, r_2)} \tag{1.8}$$

where $\overline{G}(r,s)$ and $\underline{G}(r,s)$ are the Green functions of the operator $-u'' - \frac{N-1}{r}u' + a(r)u$ in $(0,z)$ and $(z,1)$ respectively, $\overline{H}(r,s)$ and $\underline{H}(r,s)$ are their regular part (see Section 2) and finally $r_1 = \lim_{p\to+\infty} r_{1,p}$, $r_2 = \lim_{p\to+\infty} r_{2,p}$, where $r_{1,p}$ and $r_{2,p}$ are the points where $u_{1,p}$ and $u_{2,p}$ achieve there maximum.

We first show that it is possible to find $z \in (0,1)$ which verifies (1.8). Secondly we derive the existence of a suitable z_p close to z which verifies (1.7) for p large enough. We point that this point is technical and needs some careful computations.

Now we are in position to state our main result.

Let us consider denote by $G(r,s)$ the Green functions of the operator $-u'' - \frac{N-1}{r}u' + a(r)u$ in $(0,1)$ with $u(1) = 0$ and $H(r,s)$ its regular part (see Section 2 for more details).

Theorem 1.1. *Let us suppose that $\hat{r} = (\hat{r}_1, \hat{r}_2)$ is a critical point of the system*

$$\begin{cases} H_r(r_1, r_1) - \frac{H(r_1,r_1)+G(r_2,r_1)}{H(r_2,r_2)+G(r_1,r_2)} G_r(r_1, r_2) = \frac{1}{2}, \\ H_r(r_2, r_2) - \frac{H(r_2,r_2)+G(r_1,r_2)}{H(r_1,r_1)+G(r_2,r_1)} G_r(r_2, r_1) = \frac{1}{2}. \end{cases} \tag{1.9}$$

which satisfies

$$4a(\hat{r}_i) \neq \alpha_i^2, \quad \textit{for } i = 1, 2. \tag{1.10}$$

and

$$\frac{4a(\hat{r}_1)\hat{r}_2^{N-1}\alpha_2}{\alpha_1^2 - 4a(\hat{r}_1)} + \frac{4a(\hat{r}_2)\hat{r}_1^{N-1}\alpha_1}{\alpha_2^2 - 4a(\hat{r}_2)} + \hat{r}_1^{N-1}\left(\frac{H(\hat{r}_2, \hat{r}_2)}{G(\hat{r}_1, \hat{r}_2)}\alpha_2 - \alpha_1\right) \neq 0, \tag{1.11}$$

with

$$\alpha_1 = \frac{H(\hat{r}_2, \hat{r}_2) + G(\hat{r}_1, \hat{r}_2)}{H(\hat{r}_1, \hat{r}_1)H(\hat{r}_2, \hat{r}_2) - G(\hat{r}_1, \hat{r}_2)G(\hat{r}_2, \hat{r}_1)}, \tag{1.12}$$

$$\alpha_2 = \frac{H(\hat{r}_1, \hat{r}_1) + G(\hat{r}_2, \hat{r}_1)}{H(\hat{r}_1, \hat{r}_1)H(\hat{r}_2, \hat{r}_2) - G(\hat{r}_1, \hat{r}_2)G(\hat{r}_2, \hat{r}_1)}. \tag{1.13}$$

Then there exists a radial solution to (1.1) which verifies

$$u_p(r) \to \alpha_1 G(r, \hat{r}_1) + \alpha_2 G(r, \hat{r}_2) \tag{1.14}$$

uniformly on $(0,1)$.

Remark 1.2. *Note that the solution founded in Theorem 1.1 changes sign exactly once and*

$$u_p(z) = 0 \text{ if } z \text{ satisfies } \frac{G(z,\hat{r}_1)}{G(z,\hat{r}_2)} = \frac{H(\hat{r}_1,\hat{r}_1)+G(\hat{r}_2,\hat{r}_1)}{H(\hat{r}_2,\hat{r}_2)+G(\hat{r}_1,\hat{r}_2)}.$$

Remark 1.3. *It is easy to prove that α_1 and α_2 are solutions of the following linear system*

$$\sum_{j=1}^{2} \alpha_j G(r_i, r_j) = (-1)^{i+1} \quad i = 1,2. \tag{1.15}$$

Then the points $\hat{r}_1$ and $\hat{r}_2$ are critical points of the following functional,

$$F(r_1, r_2) = \sum_{j=1}^{2} (-1)^{j+1} \alpha_j r_j^{N-1}, \ r = (r_1, r_2) \in [0,1] \times [0,1]. \tag{1.16}$$

The paper is organized as follows: In Section 2 we prove some properties of the Green function of the operator $\mathcal{L}u = -u'' - \frac{N-1}{r}u' + a(r)u$ which will be used in the paper. In Section 3 we study the asymptotic behaviour of the solution $u_{2,p}$ as $p \to +\infty$. In Section 4 we prove Theorem 1.1 and in Section 5 we give an example of a function $a(r)$ which verifies the assumption of Theorem 1.1.

2 Preliminaries

We start this section with some remarks on the Green function of the operator $\mathcal{L}u = -u'' - \frac{N-1}{r}u' + a(r)u$, $u \in H^1(0,1) : u(1) = 0$. It is likely that most of the main properties claimed in this section are known, but we are not able to provide any reference. The argument is quite elementary. Let $N \geq 3$ and introduce the function

$$\begin{aligned} \Gamma(r,s) &= \frac{s^{N-1}}{2(N-2)} \left(r^{2-N} - s^{2-N} - \left| r^{2-N} - s^{2-N} \right| \right) \\ &= \begin{cases} 0 & \text{if } r < s \\ \frac{s^{N-1}}{N-2} \left(r^{2-N} - s^{2-N} \right) & \text{if } r > s \end{cases} \end{aligned} \tag{2.1}$$

for $r, s \in (0,1)$. It is easy to check that $\Gamma(r,s)$ verifies

$$\begin{cases} -\Gamma_{rr}(r,s) - \frac{N-1}{r}\Gamma_r(r,s) = \delta_s(r) & r \in (0,1) \\ \Gamma_r(0,s) = 0 \end{cases}$$

in the sense of distribution. Here $\delta_s(r)$ is the Dirac function centered at s. Then we have the following decomposition for the Green function,

$$G(r,s) = \Gamma(r,s) + H(r,s) \tag{2.2}$$

where $H(r,s)$ is the solution of the problem

$$\begin{cases} -H_{rr}(r,s) - \frac{N-1}{r} H_r(r,s) + a(r)H(r,s) = -a(r)\Gamma(r,s) & r \in (0,1) \\ H_r(0,s) = 0 \\ H(1,s) = \frac{1}{N-2}\left(s - s^{N-1}\right). \end{cases} \tag{2.3}$$

$H(r,s)$ is called the *regular part* of the Green function; it is not difficult to show that $H(r,s)$ and $G(r,s)$ are uniformly bounded in $(0,1)$. Since $\Gamma(s,s) = 0$ we will write $G(r,r) = H(r,r)$. We have the following

Lemma 2.1. *The Green function $G(r,s)$ satisfies*

$$G(r,s)r^{N-1} = G(s,r)s^{N-1}, \tag{2.4}$$

$$G(r,s) < H(s,s) \text{ for } r \neq s, \tag{2.5}$$

Proof. Proof of (2.4): The Green function $G(r,s)$ satisfies the equation

$$\begin{cases} -G_{rr}(r,s) - \frac{N-1}{r} G_r(r,s) + a(r)G(r,s) = \delta_s(r) & r \in (0,1) \\ G_r(0,s) = 0 \\ G(1,s) = 0. \end{cases} \tag{2.6}$$

Multiplying (2.6) by $G(r,t)r^{N-1}$ and integrating we get

$$\int_0^1 \left[G_r(r,s)G_r(r,t) + a(r)G(r,s)G(r,t)\right] r^{N-1} dr = s^{N-1} G(s,t) \tag{2.7}$$

Writing down the equation satisfied by $G(r,t)$ and multiplying it by $G(r,s)r^{N-1}$ we have the claim.

Proof of (2.5): let $r < s$. Then the Green function satisfies

$$-G_{rr}(r,s) - \frac{N-1}{r} G_r(r,s) + a(r)G(r,s) = 0 \tag{2.8}$$

Integrating (2.8) we get

$$G_r(r,s)r^{N-1} = \int_0^r a(t)t^{N-1}G(t,s)dt > 0 \tag{2.9}$$

From (2.9) we derive $G(r,s) < G(s,s) = H(s,s)$; arguing in the same way for $r > s$ we have the claim.

∎

Corollary 2.2. *The following identities hold*

$$\left(G_r(r,s) + \frac{N-1}{r}G(r,s)\right) r^{N-1} = G_s(s,r)s^{N-1}, \tag{2.10}$$

$$H_r(r,r) + \frac{N-1}{r}H(r,r) - 1 = H_s(r,r), \tag{2.11}$$

Proof. See [G2].

■

Next proposition states an important decomposition property of the Green function. It will be greatly used in all the paper.

Proposition 2.3. *The following decomposition holds*

$$G(r,s) = \begin{cases} A(r)B(s) & \text{if } 0 \le r \le s \\ \widetilde{A}(r)\widetilde{B}(s) & \text{if } s \le r \le 1 \end{cases} \tag{2.12}$$

for some positive smooth function $A, B, \widetilde{A}, \widetilde{B}$.

Proof. Let $0 \le r < s$ and consider the function

$$\overline{G}(r,q,s) = G(r,q) - \frac{G(P,q)}{G(P,s)} G(r,s) \quad \text{with } 0 \le r, q < P < s. \tag{2.13}$$

we have that $\overline{G}(r,q,s)$ satisfies

$$\begin{cases} -\overline{G}_{rr}(r,s) - \frac{N-1}{r}\overline{G}_r(r,s) + a(r)\overline{G}(r,s) = \delta_q(r) & r \in (0,P) \\ \overline{G}_r(0,q) = 0 \\ \overline{G}(P,q) = 0. \end{cases} \tag{2.14}$$

Hence $\overline{G}(P,q)$ is the Green function of the operator $\mathcal{L}u = -u'' - \frac{N-1}{r}u' + a(r)u, \quad r \in (0,P)$. From the uniqueness of the Green function it follows that $\overline{G}(P,q)$ is independent of s and then

$$0 = \frac{\partial}{\partial s}\overline{G}(P,q) = -G(P,q)\frac{\partial}{\partial s}\left(\frac{G(r,s)}{G(P,s)}\right) \tag{2.15}$$

and then

$$G(r,s) = A(r)G(P,s) \tag{2.16}$$

Setting $B(s) = G(P,s)$ the claim follows for $0 \le r \le s$. The case $\le s \le r \le 1$ is similar.

■

Corollary 2.4. *The following identities hold*

$$A(r)B'(r) - \widetilde{A}(r)\widetilde{B}'(r) = -1, \tag{2.17}$$

$$A'(r)B(r) - \widetilde{A}'(r)\widetilde{B}(r) = 1 \tag{2.18}$$

$$A'(r)B(s)r^{N-1} = \widetilde{A}(s)\left(\widetilde{B}'(r) - \frac{N-1}{r}\widetilde{B}(r)\right)s^{N-1}, \ \textit{for } r \le s. \tag{2.19}$$

Proof. Let us prove (2.17). By Proposition 2.3 we get, for $r < s$,

$$A(r)B(s) = G(r,s) = H(r,s)$$

and then

$$A(r)B'(s) = H_s(r,s) \Rightarrow A(r)B'(r) = H_s(r,r).$$

On the other hand, for $r > s$

$$\widetilde{A}(r)\widetilde{B}(s) = G(r,s) = \Gamma(r,s) + H(r,s) = \frac{s^{N-1}}{N-2}\left(r^{2-N} - s^{2-N}\right) + H(r,s)$$

and then

$$\widetilde{A}(r)\widetilde{B}'(s) = \frac{1}{N-2}\left((N-1)s^{N-2}r^{2-N} - 1\right) + H_s(r,s).$$

So

$$\widetilde{A}(r)\widetilde{B}'(r) = 1 + H_s(r,s) = 1 + A(r)B'(r)$$

and (2.17) follows.

Proof of (2.18) is analogous. Finally (2.19) follows by (2.10) and (2.4). ■

Proposition 2.5. *Let us denote by* $\overline{G}(r,s) = \Gamma(r,s) + \overline{H}(r,s)$ *the Green function of the operator* $\mathcal{L}u = -u'' - \frac{N-1}{r}u' + a(r)u, \quad r \in (0,z)$.

If $\overline{r} \in (0,z)$ *is a critical point of the function*

$$F(r) = \frac{\overline{H}(r,r)}{r^{N-1}} \tag{2.20}$$

we have that

$$H_{rr}(\overline{r},\overline{r}) + H_{rs}(\overline{r},\overline{r}) = \frac{4a(\overline{r})\overline{H}(\overline{r},\overline{r})^2 - 1}{4H(\overline{r},\overline{r})}. \tag{2.21}$$

Proof. Let us recall (see (3.1) and (3.2) in [G2]) that the assumption that $\overline{r}$ is a critical point of F is equivalent to

$$H_r(\overline{r},\overline{r}) = \frac{1}{2} \tag{2.22}$$

Using Proposition 2.3 we see that (2.22) is equivalent to

$$A'(\overline{r})B(\overline{r}) = \frac{1}{2} \tag{2.23}$$

and

$$\begin{aligned} &H_{rr}(\overline{r},\overline{r}) + H_{rs}(\overline{r},\overline{r}) = A''(\overline{r})B(\overline{r}) + A'(\overline{r})B'(\overline{r}) = \\ &-\frac{N-1}{r}A'(\overline{r})B(\overline{r}) + a(\overline{r})A(\overline{r})B(\overline{r}) + A'(\overline{r})B'(\overline{r}) = \\ &-\frac{N-1}{2r} + a(\overline{r})A(\overline{r})B(\overline{r}) + \frac{A(\overline{r})B'(\overline{r})}{2A(\overline{r})B(\overline{r})} = \\ &\frac{4a(\overline{r})A^2(\overline{r})B^2(\overline{r}) - 1}{4A(\overline{r})B(\overline{r})} = \frac{4a(\overline{r})H(\overline{r},\overline{r})^2 - 1}{4H(\overline{r},\overline{r})}. \end{aligned} \tag{2.24}$$

Next theorem is an improving of the main result of [G2].

Theorem 2.6. *Let us denote by* $\overline{G}(r,s)=\Gamma(r,s)+\overline{H}(r,s)$ *the Green function of the operator* $\mathcal{L}u=-u''-\frac{N-1}{r}u'+a(r)u, \quad r\in(0,z)$.

If there exists a critical point $\overline{r}\in(0,z)$ *of the function*

$$F(r)=\frac{\overline{H}(r,r)}{r^{N-1}} \tag{2.25}$$

satisfying

$$4a(\overline{r})\overline{H}(\overline{r},\overline{r})^2\neq 1 \tag{2.26}$$

then there exist a solution to the problem

$$\begin{cases} -u_p''-\frac{N-1}{r}u_p'+a(r)u_p=u_p^p & \text{for } 0<r<z,\\ u_p>0 & \text{for } 0<r<z,\\ u_p'(0)=0, u_p(z)=0. \end{cases} \tag{2.27}$$

verifying

$$u_p(r)\to\frac{G(r,\overline{r})}{H(\overline{r},\overline{r})} \quad in\ H_0^1(0,z)\cap C^1\left([0,z]\setminus\{\overline{r}\}\right). \tag{2.28}$$

Proof. In [G2] it was proved that if $\overline{r}$ is a nondegenerate critical point of the function F there exists a solution to (2.27) satisfying (2.28). Let us recall (see [G2]) that the nondegeneracy condition is

$$H_{rr}(\overline{r},\overline{r})+H_{rs}(\overline{r},\overline{r})\neq 0 \tag{2.29}$$

Using Proposition 2.5 we have the claim;

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3 Analysis of the solution in the annulus

Let us consider the following minimization problem

$$I_p=\inf_{u\in H_r^1(\Omega)}\frac{\left(\int_\Omega|\nabla u|^2+a(|x|)u^2\right)}{\left(\int_\Omega|u|^{p+1}\right)^{\frac{2}{p+1}}} \tag{3.1}$$

where $\Omega=\left\{x\in\mathbb{R}^N:0<a\leq|x|\leq b\right\}$, $H_r^1(\Omega)=\left\{u\in H_0^1(\Omega):u=u(|x|)\right\}$ and $0\leq a(|x|)\leq C$. Up to a constant the minimum $u_p=u_p(r)$ of (3.1) satisfies

$$\begin{cases} -u_p''-\frac{N-1}{r}u_p'+a(r)u_p=u_p^p & \text{for } a<r<b,\\ u_p>0 & \text{for } a<r<b,\\ u_p(a)=u_p(b)=0. \end{cases} \tag{3.2}$$

The aim of this section is to study the asymptotic behavior of the solution u_p as $p\to+\infty$. The proof of some of these results is the same of the case $a(r)\equiv 0$ considered in [G1] and it will be omitted.

Let us start with the following lemma;

Lemma 3.1. *We have that,*

$$\int_a^b u_p'(r)^2 dr \le C, \tag{3.3}$$

$$u_p(r) \to \frac{\underline{G}(r,r_0)}{\underline{G}(r_0,r_0)} \ \textit{in}\ H_0^1(a,b) \cap C^1\left([a,b]\setminus\{r_0\}\right), \tag{3.4}$$

$$\underline{H}_r(r_0,r_0) = \frac{1}{2}. \tag{3.5}$$

where $\underline{G}(r,s)$ *is the Green function of the operator* $-u'' - \frac{N-1}{r}u' + a(r)u$ *in* (a,b).

Proof. Proof of (3.3) is the same as in [G1].

Let us prove (3.4); first we remark that integrating (3.2) one gets from (3.3)

$$|u_p'(r)| \le C \quad \text{for any } r \in [a,b] \tag{3.6}$$

Hence there exists $\phi \in H_0^1(a,b)$ such that

$$u_p \to \phi \quad \text{in } H_0^1(a,b). \tag{3.7}$$

Let us show that $||\phi||_\infty = 1$.

Multiplying (3.2) by $e_1 r^{N-1}$ where e_1 is the first eigenfunction of the operator $-u'' - \frac{N-1}{r}u'$ in (a,b) we get

$$\lambda_1 \int_a^b u_p e_1 \le \int_a^b u_p^p e_1 \tag{3.8}$$

where λ_1 is the eigenvalue associated to e_1. From (3.8) we get that $||u_p||_\infty^{p-1} \ge \lambda_1$ and then $||\phi||_\infty \ge 1$.

On the other hand if $||\phi||_\infty > 1$ we get that $||u_p||_\infty > 1$ for p large. Moreover, since $|u_p'(r)| \le C$, we derive that there exists a set of positive measure such that $u_p > 1$. Then

$$\int_a^b u_p^{p+1} \to +\infty \tag{3.9}$$

and this is a contradiction with (3.3). So $||\phi||_\infty = 1$.

Recalling that u_p minimizes (3.1), $u_p \to \phi$ in $H_0^1(a,b)$ and $||\phi||_\infty = 1$ we get

$$I_p = \min_{\substack{u\in H_0^1(a,b)\\ \int_a^b u^{p+1}=1}} \int_a^b \left(|u'|^2 + a(r)u^2\right) r^{N-1} \to I_\infty = \min_{\substack{u\in H_0^1(a,b)\\ ||u||_\infty=1}} \int_a^b \left(|u'|^2 + a(r)u^2\right) r^{N-1} =$$

$$\int_a^b \left(|\phi'|^2 + a(r)\phi^2\right) r^{N-1}. \tag{3.10}$$

Set

$$A = \{r \in (a,b) : \phi(r) = 1\} \tag{3.11}$$

We claim that A is a singleton. By contradiction let us suppose that card$A \geq 2$. This implies that

$$I_\infty = \min_{\substack{u \in H_0^1(a,b) \\ ||u||_\infty = 1 \\ \text{card}_{A \geq 2}}} \int_a^b \left(|u'|^2 + a(r)u^2\right) r^{N-1} \tag{3.12}$$

Now let us denote by ψ the function which achieves I_∞ with $A = \{r_1, r_2\}$. We have that ψ solves

$$\begin{cases} -\psi'' - \frac{N-1}{r}\psi' + a(r)\psi = 0 & \text{in } (a, r_1) \cup (r_1, r_2) \cup (r_2, b), \\ \psi > 0 & \text{in } a < r < b, \\ \psi(r_1) = \psi(r_2) = 1, \ \psi \in H_0^1(a,b) \end{cases} \tag{3.13}$$

A straightforward computation gives that

$$\psi(r) = \alpha \underline{G}(r, r_1) + \beta \underline{G}(r, r_2) \tag{3.14}$$

where $\alpha, \beta \in \mathbb{R}$ are chosen so that $\psi(r_1) = \psi(r_2) = 1$. Actually we have that

$$\alpha = \frac{\underline{H}(r_2, r_2) - \underline{G}(r_1, r_2)}{\underline{H}(r_1, r_1)\underline{H}(r_2, r_2) - \underline{G}(r_1, r_2)\underline{G}(r_2, r_1)}, \tag{3.15}$$

$$\beta = \frac{\underline{H}(r_1, r_1) - \underline{G}(r_2, r_1)}{\underline{H}(r_1, r_1)\underline{H}(r_2, r_2) - \underline{G}(r_1, r_2)\underline{G}(r_2, r_1)}. \tag{3.16}$$

We have that

$$\begin{aligned} I_\infty &= \int_a^b \left(|\psi'|^2 + a(r)\psi^2\right) r^{N-1} = \\ &\alpha r_1^{N-1} \left(\alpha \underline{H}(r_1, r_1) + \beta \underline{G}(r_1, r_2)\right) + \beta r_2^{N-1} \left(\alpha \underline{G}(r_2, r_1) + \beta \underline{H}(r_2, r_2)\right) = \\ &\frac{\underline{H}(r_2, r_2) r_1^{N-1} + \underline{H}(r_1, r_1) r_2^{N-1} - 2\underline{G}(r_1, r_2) r_1^{N-1}}{\underline{H}(r_1, r_1)\underline{H}(r_2, r_2) - \underline{G}(r_1, r_2)\underline{G}(r_2, r_1)} > \frac{r_1^{N-1}}{\underline{H}(r_1, r_1)} \end{aligned} \tag{3.17}$$

since $\underline{H}(r_1, r_1) - \underline{G}(r_2, r_1) > 0$. Finally choosing $\bar{\phi} = \frac{\underline{G}(r, r_3)}{\underline{H}(r_0, r_3)}$ with r_3 satisfying

$$\frac{r_3^{N-1}}{\underline{H}(r_3, r_3)} = \min_{r \in (a,b)} \frac{r^{N-1}}{\underline{H}(r, r)} \tag{3.18}$$

we obtain

$$I_\infty \leq \int_a^b \left(|\bar{\phi}'|^2 + a(r)\bar{\phi}^2\right) r^{N-1} = \frac{r_3^{N-1}}{\underline{H}(r_3, r_3)} \tag{3.19}$$

which gives a contradiction with (3.17). Hence $A=\{r_0\}$ is a singleton and ϕ solves

$$\begin{cases} -\phi'' - \frac{N-1}{r}\phi' + a(r)\phi' = 0 & \text{in } (a,r_0)\cup(r_0,b), \\ \phi(r_0)=1,\ \phi\in H_0^1(a,b). \end{cases} \tag{3.20}$$

Then $\phi(r)=\frac{\underline{G}(r,r_0)}{\underline{H}(r_0,r_0)}$ which proves (3.4).

Finally we prove (3.5). Let us recall that

$$\inf_{u\in H_r^1(\Omega)} \frac{\left(\int_\Omega |\nabla u|^2 + a(|x|)^2\right)}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}} = \left(\int_a^b \left(u_p'(r)^2 + a(r)u_p^2\right) r^{N-1} dr\right)^{1-\frac{2}{p+1}} =$$

$$\frac{\int_a^b \left(\underline{G}'(r,r_0)^2 + a(r)\underline{G}(r,r_0)^2\right) r^{N-1} dr}{\underline{G}(r_0,r_0)^2} + o(1) = \frac{\underline{G}(r_0,r_0) r_0^{N-1}}{\underline{G}(r_0,r_0)^2} + o(1) =$$

$$\frac{r_0^{N-1}}{\underline{H}(r_0,r_0)} + o(1) \tag{3.21}$$

Hence r_0 minimizes the function $F(r)=\frac{r^{N-1}}{\underline{H}(r,r)}$ and then $F'(r_0)=0$. Computing the derivative of F we get

$$F'(r) = \frac{\frac{N-1}{r}\underline{H}(r,r) - \underline{H}_r(r,r) - \underline{H}_s(r,r)}{\underline{H}(r,r)^2} r^{N-1} \Rightarrow$$

$$\frac{N-1}{r_0}\underline{H}(r_0,r_0) = \underline{H}_r(r_0,r_0) + \underline{H}_s(r_0,r_0), \tag{3.22}$$

and recalling (2.11) we have the claim.

∎

4 Existence of the nodal solution

In this section we construct a nodal solution to (1.1) which changes sign exactly once. We assume that $r_1, r_2 \in (0,1)$ satisfy

$$H_r(r_1,r_1) - \frac{H(r_1,r_1)+G(r_2,r_1)}{H(r_2,r_2)+G(r_1,r_2)} G_r(r_1,r_2) = \frac{1}{2}, \tag{4.1}$$

and

$$H_r(r_2,r_2) - \frac{H(r_2,r_2)+G(r_1,r_2)}{H(r_1,r_1)+G(r_2,r_1)} G_r(r_2,r_1) = \frac{1}{2}. \tag{4.2}$$

We have the following

Lemma 4.1. *Let us assume that r_1 and r_2 satisfy (4.1) and (4.2). Then*

$$\frac{A'(r_1)}{A(r_1)} = \frac{H(r_2,r_2)+G(r_1,r_2)}{H(r_1,r_1)H(r_2,r_2) - G(r_1,r_2)G(r_2,r_1)} \tag{4.3}$$

and

$$\frac{A'(r_2)}{A(r_2)} = \frac{H(r_1,r_1)+G(r_2,r_1)}{H(r_1,r_1)H(r_2,r_2)-G(r_1,r_2)G(r_2,r_1)} \tag{4.4}$$

Proof. From (4.1) we derive

$$\begin{aligned}
&H_r(r_1,r_1) - \frac{H(r_1,r_1)+G(r_2,r_1)}{H(r_2,r_2)+G(r_1,r_2)}G_r(r_1,r_2) = \frac{1}{2} \Leftrightarrow \\
&A'(r_1)B(r_1) - \frac{A(r_1)B(r_1)+\widetilde{A}(r_2)\widetilde{B}(r_1)}{A(r_2)B(r_2)+A(r_1)B(r_2)}A'(r_1)B(r_2) = \frac{1}{2} \Leftrightarrow \\
&A'(r_1) = \frac{A(r_1)+A(r_2)}{2\Big(A(r_2)B(r_1)-\widetilde{A}(r_2)\widetilde{B}(r_1)\Big)} \Leftrightarrow \qquad (4.5)\\
&\frac{A'(r_1)}{A(r_1)} = \frac{A(r_1)B(r_2)+A(r_2)B(r_2)}{2\Big(A(r_1)B(r_1)A(r_2)B(r_2)-A(r_1)B(r_2)\widetilde{A}(r_2)\widetilde{B}(r_1)\Big)} = \\
&\frac{H(r_2,r_2)+G(r_1,r_2)}{H(r_1,r_1)H(r_2,r_2)-G(r_1,r_2)G(r_2,r_1)}.
\end{aligned}$$

which proves (4.3). In the same way we get (4.4).

■

Now we decompose the interval $[0,1]$ in the following way

$$[0,1] = [0,z] \cup [z,1],$$

where z will be chosen later. In next proposition we show the existence of a positive solution of the problem

$$\begin{cases} -u'' - \frac{N-1}{r}u' + a(r)u = u^p, & r \in (0,z) \\ u > 0,\ u'(0) = 0,\ u(z) = 0. \end{cases} \tag{4.6}$$

Proposition 4.2. *Let us assume (4.1) and*

$$4a(r_1) \neq \left(\frac{H(r_2,r_2)+G(r_1,r_2)}{H(r_1,r_1)H(r_2,r_2)-G(r_1,r_2)G(r_2,r_1)}\right)^2 \tag{4.7}$$

Moreover let us choose $z \in (r_1,r_2)$ satisfying

$$\frac{G(z,r_1)}{G(z,r_2)} = \frac{H(r_1,r_1)+G(r_2,r_1)}{H(r_2,r_2)+G(r_1,r_2)} \tag{4.8}$$

Then, for p large enough there exists a solution $u_{1,p}$ to (4.6) which satisfies

$$u_{1,p}(r) \to \frac{\overline{G}(r,r_1)}{\overline{H}(r_1,r_1)} \quad \text{in } C([0,z]) \cap C^1\left([0,z] \setminus \{r_1\}\right). \tag{4.9}$$

where $\overline{G}(r,s)$ is the Green function of the operator $\mathcal{L}u = -u'' - \frac{N-1}{r}u + a(r)u$ with $u \in \mathcal{C} = \big\{u \in C^2(B_z) : u'(0) = u(z) = 0\big\}$.

Proof. First we show that there exists $z \in (r_1, r_2)$ verifying (4.8). Using (2.5) we have that

$$\lim_{z \to r_1} \frac{G(z,r_1)}{G(z,r_2)} = \frac{H(r_1,r_1)}{G(r_1,r_2)} > \frac{H(r_1,r_1) + G(r_2,r_1)}{H(r_2,r_2) + G(r_1,r_2)}, \tag{4.10}$$

and

$$\lim_{z \to r_2} \frac{G(z,r_1)}{G(z,r_2)} = \frac{G(r_2,r_1)}{H(r_2,r_2)} < \frac{H(r_1,r_1) + G(r_2,r_1)}{H(r_2,r_2) + G(r_1,r_2)} \tag{4.11}$$

The continuity of the Green function gives the existence of $z \in (r_1, r_2)$ verifying (4.8).

Recalling Theorem 2.6 we have that there exists a solution to (4.6) if the function $\frac{\overline{H}(r,r)}{r^{N-1}}$ admits a critical point r_1 verifying $4a(r_1) \neq \frac{1}{\overline{H}(r_1,r_1)^2}$. Here $\overline{H}(r,r)$ is the regular part of the Green function $\overline{G}(r,s)$. Let us show that the condition (4.1) implies that r_1 is a critical point of the function $\frac{\overline{H}(r,r)}{r^{N-1}}$.

To do this, let us point out that for any $r_2 > z$, the function

$$\overline{G}(r,s) = G(r,s) - \frac{G(z,s)}{G(z,r_2)} G(r,r_2) \tag{4.12}$$

is the Green function of the operator $\mathcal{L}u$ with $u \in \mathcal{C}$.

Then (2.22) of Theorem 2.6 becomes

$$\overline{H}_r(r_1,r_1) = \frac{1}{2} \Leftrightarrow \; H_r(r_1,r_1) - \frac{G(z,r_1)}{G(z,r_2)} G_r(r_1,r_2) = \frac{1}{2} \Leftrightarrow$$
$$H_r(r_1,r_1) - \frac{H(r_1,r_1) + G(r_2,r_1)}{H(r_2,r_2) + G(r_1,r_2)} G_r(r_1,r_2) = \frac{1}{2}, \tag{4.13}$$

and

$$4a(r_1) \neq \frac{1}{\overline{H}(r_1,r_1)^2} \Leftrightarrow 4a(r_1) \neq \frac{1}{\left(H(r_1,r_1) - \frac{G(z,r_1)}{G(z,r_2)} G(r_1,r_2)\right)^2} =$$
$$\left(\frac{H(r_2,r_2) + G(r_1,r_2)}{H(r_1,r_1)H(r_2,r_2) - G(r_1,r_2)G(r_2,r_1)}\right)^2. \tag{4.14}$$

So Theorem 2.6 applies and we have the existence of a solution to (4.6).

∎

Proposition 4.3. *Let us assume that z satisfies (4.8). Then the solution $u_{2,p}$ of*

$$\begin{cases} -u'' - \frac{N-1}{r} u' + a(r)u = u^p, & r \in (z,1) \\ u > 0, \; u(z) = 0, \; u(1) = 0, \end{cases} \tag{4.15}$$

verifies

$$u_{2,p}(r) \to \frac{\underline{G}(r,r_2)}{\underline{H}(r_2,r_2)} \quad \text{in } H_0^1(z,1) \cap C^1\left([z,1] \setminus \{r_1\}\right). \tag{4.16}$$

where $\underline{G}(r,s)$ is the Green function of the operator $\mathcal{L}u = -u'' - \frac{N-1}{r}u + a(r)u$ with $u \in C^2(z,1) \cap C_0([z,1])$. Finally r_2 satisfies (4.2).

Proof. The existence of the solution and (4.16) follows by Lemma 3.1. In order to finish the proof we have to show that r_2 satisfies (3.5). To do this let us remark that the function

$$\underline{G}(r,s) = G(r,s) - \frac{G(z,s)}{G(z,r_1)} G(r,r_1) \tag{4.17}$$

is the Green function of the operator $\mathcal{L}u$ with $u \in C^2(z,1) \cap C_0([z,1])$. From (4.17) we derive

$$\underline{H}(r,r_2) = H(r,r_2) - \frac{G(z,r_2)}{G(z,r_1)} G(r,r_1) \tag{4.18}$$

and differentiating (4.18) we have

$$\begin{aligned} &\underline{H}_r(r_2,r_2) = \frac{1}{2} \Leftrightarrow \ H_r(r_2,r_2) - \frac{G(z,r_2)}{G(z,r_1)} G_r(r_2,r_1) = \frac{1}{2} \Leftrightarrow \\ &H_r(r_2,r_2) - \frac{H(r_2,r_2)+G(r_1,r_2)}{H(r_1,r_1)+G(r_2,r_1)} G_r(r_2,r_1) = \frac{1}{2}, \end{aligned} \tag{4.19}$$

Hence r_2 satisfies (4.2).

∎

Now we look for a solution u_p to (1.1) as

$$u_p = \begin{cases} u_{1,p}(r) & \text{for } 0 \le r \le z \\ -u_{2,p}(r) & \text{for } z \le r \le 1, \end{cases} \tag{4.20}$$

where $u_{1,p}$ and $u_{2,p}$ are the solutions given by Propositions 4.2 and 4.3 respectively.

Of course we have that u_p is a solution to (4.6) in $[0,1]$ if

$$u'_{1,p}(z) = -u'_{2,p}(z) \tag{4.21}$$

Next lemma states that (4.21) is true as $p \to \infty$.

Lemma 4.4. *We have that*

$$\lim_{p\to\infty} u'_{1,p}(z) = -\lim_{p\to\infty} u'_{2,p}(z) \tag{4.22}$$

Proof. By (4.9) and (4.12) we have that

$$\lim_{p\to\infty} u'_{1,p}(z) = \frac{\overline{G}_r(z,r_1)}{\overline{H}(r_1,r_1)} = \frac{G_r(z,r_1) - \frac{G(z,r_1)}{G(z,r_2)} G_r(z,r_2)}{H(r_1,r_1) - \frac{G(z,r_1)}{G(z,r_2)} G(r_1,r_2)} \tag{4.23}$$

and since z verifies (4.8) we get

$$\begin{aligned} \lim_{p\to\infty} u'_{1,p}(z) &= \frac{G_r(z,r_1) - \frac{H(r_1,r_1)+G(r_2,r_1)}{H(r_2,r_2)+G(r_1,r_2)} G_r(z,r_2)}{H(r_1,r_1) - \frac{H(r_1,r_1)+G(r_2,r_1)}{H(r_2,r_2)+G(r_1,r_2)} G(r_1,r_2)} \\ &= \frac{G_r(z,r_1)(H(r_2,r_2)+G(r_1,r_2)) - (H(r_1,r_1)+G(r_2,r_1))G_r(z,r_2)}{H(r_1,r_1)H(r_2,r_2) - G(r_1,r_2)G(r_2,r_1)} \end{aligned} \tag{4.24}$$

Arguing in the same way, by (3.4) and (4.17) we have

$$
\begin{aligned}
&-\lim_{p\to\infty} u'_{2,p}(z) = -\frac{\underline{G}_r(z,r_2)}{\underline{H}(r_2,r_2)} = -\frac{G_r(z,r_2) - \frac{G(z,r_2)}{G(z,r_1)}G_r(z,r_1)}{H(r_2,r_2) - \frac{G(z,r_2)}{G(z,r_1)}G(r_2,r_1)} \\
&-\frac{G_r(z,r_2) - \frac{H(r_2,r_2)+G(r_1,r_2)}{H(r_1,r_1)+G(r_2,r_1)}G_r(z,r_2)}{H(r_2,r_2) - \frac{H(r_2,r_2)+G(r_1,r_2)}{H(r_1,r_1)+G(r_2,r_1)}G(r_1,r_2)} \\
&= -\frac{G_r(z,r_2)(H(r_1,r_1)+G(r_2,r_1)) - (H(r_2,r_2)+G(r_1,r_2))G_r(z,r_1)}{H(r_1,r_1)H(r_2,r_2) - G(r_1,r_2)G(r_2,r_1)}
\end{aligned}
\tag{4.25}
$$

which gives the claim.

■

In order to prove our theorem we have to show that (4.22) holds for p large. This is the final step of our proof and it requires some work.

Let us set

$$z(t) = z + t, t \in [-\epsilon, \epsilon] \text{ for a suitable positive } \epsilon$$

and point out that the nondegeneracy of the critical point r_1 of the functional

$$
\begin{aligned}
\frac{\overline{H}(r,r)}{r^{N-1}} &= \frac{1}{r^{N-1}}\left(H(r,r) - \frac{G(z,r)}{G(z,r_2)}G(r,r_2)\right) \\
&= \frac{1}{r^{N-1}}\left(H(r,r) - \frac{G(z,r)}{A(z)}A(r)\right)
\end{aligned}
\tag{4.26}
$$

implies that also the functionals

$$\frac{\overline{H}_t(r,r)}{r^{N-1}} = \frac{1}{r^{N-1}}\left(H(r,r) - \frac{G(z(t),r)}{G(z(t),r_2)}G(r,r_2)\right) \tag{4.27}$$

and

$$\frac{\underline{H}_t(r,r)}{r^{N-1}} = \frac{1}{r^{N-1}}\left(H(r,r) - \frac{G(z(t),r)}{G(z(t),r_1)}G(r,r_1)\right) \tag{4.28}$$

admit a nondegenerate critical point $r_1(t) \to r_1$ as $t \to 0$. Moreover since $\overline{H}_t(r,r)$ is the Robin function in $(0, z(t))$, by Theorem 2.6 for t small enough there exists a solution $u_{1,t,p}(r)$ to

$$
\begin{cases}
-u'' - \frac{N-1}{r}u' + a(r)u = u^p, & r \in (0, z(t)) \\
u > 0,\ u'(0) = 0,\ u(z(t)) = 0,
\end{cases}
\tag{4.29}
$$

On the other hand let us denote by $u_{2,t,p}(r)$ the solution of the problem

$$
\begin{cases}
-u'' - \frac{N-1}{r}u' + a(r)u = u^p, & r \in (z(t), 1) \\
u > 0,\ u(z(t)) = 0,\ u(1) = 0,
\end{cases}
\tag{4.30}
$$

considered in Section 3 and set by $r_2(t)$ the limit of its maximum point.

Finally let us consider the function

$$f_p(t) = u'_{1,t,p}(z(t)) + u'_{2,t,p}(z(t)). \tag{4.31}$$

Lemma 3.1 and Proposition 4.2 tell us that

$$f_p(t) \to f_\infty(t) = \frac{G_r(z(t), r_1(t)) - \frac{G(z(t), r_1(t))}{A(z(t))} A'(z(t))}{H(r_1(t), r_1(t)) - \frac{G(z(t), r_1(t))}{A(z(t))} A(r_1(t))} + \frac{G_r(z(t), r_2(t)) - \frac{G(z(t), r_2(t))}{A(z(t))} A'(z(t))}{H(r_2(t), r_2(t)) - \frac{G(z(t), r_2(t))}{A(z(t))} A(r_2(t))} \quad \text{in } C^1(-\epsilon, \epsilon). \tag{4.32}$$

Since $f_\infty(0) = 0$ (see Lemma 4.4) if we show that $f'_\infty(0) \neq 0$ then there exists t close to zero such that $f_p(t) = 0$ and this gives the claim. Proof of $f'_\infty(0) \neq 0$ is quite tedious and needs a lot of computation. Let us start with the following

Lemma 4.5. *Let us assume that r_1, r_2 satisfy (4.1), (4.2), (4.7) and*

$$4a(r_2) \neq \left(\frac{H(r_1, r_1) + G(r_2, r_1)}{H(r_2, r_2)H(r_1, r_1) - G(r_2, r_1)G(r_1, r_2)} \right)^2 \tag{4.33}$$

Then the functions $r_1(t)$, $r_2(t)$ are differentiable and

$$r'_1(t) = \frac{2A'(r_1(t))^2 A(r_1(t)) \widetilde{B}(r_1(t))}{\left(A'(r_1(t))^2 - a(r_1(t))A(r_1(t))^2\right) A(z(t)) \widetilde{B}(z(t))} > 0, \tag{4.34}$$

$$r'_2(t) = \frac{2\widetilde{A}'(r_2(t))^2 \widetilde{A}(r_2(t)) B(r_2(t))}{\left(\widetilde{A}'(r_2(t))^2 - a(r_2(t))\widetilde{A}(r_2(t))^2\right) \widetilde{A}(z(t)) B(z(t))} > 0, \tag{4.35}$$

Proof. Let us recall that conditions $\overline{H}_r(r_1, r_1) = \frac{1}{2}$ and $4a(r_1) \neq \frac{1}{\overline{H}(r_1, r_1)^2}$ imply that r_1 is a nondegenerate critical point of the function $F(r) = \frac{\overline{H}(r,r)}{r^{N-1}}$ (see Proposition 2.5).

Since $\overline{H}(r, r) = H(r, r) - \frac{G(z,r)}{G(z,r_2)} G(r, r_2)$ we have that (4.1) and (4.7) imply that r_1 is a nondegenerate critical point for the functional $\frac{H(r,r) - \frac{G(z,r)}{G(z,r_2)} G(r,r_2)}{r^{N-1}}$.

So, applying the implicit function theorem to

$$F(t, r) = \frac{H_r(r, r) - \frac{G(z(t),r)}{G(z(t),r_2)} G_r(r, r_2)}{r^{N-1}}$$

we get that the function $r_1(t)$ is differentiable and it satisfies

$$\begin{aligned}
&\frac{d}{dr}F(t,r)=0 \Leftrightarrow\\
&H_r(r_1(t),r_1(t))-\frac{G(z(t),r_1(t))}{G(z(t),r_2(t))}G_r(r_1(t),r_2(t))=\frac{1}{2}\Leftrightarrow\\
&H_r(r_1(t),r_1(t))-\frac{G(z(t),r_1(t))}{A(z(t))}A'(r_1(t))=\frac{1}{2}\Leftrightarrow\\
&A'(r_1(t))B(r_1(t))-\frac{\widetilde{A}(z(t)\widetilde{B}(r_1(t))}{A(z(t))}A'(r_1(t))=\frac{1}{2}\Leftrightarrow \qquad (4.36)\\
&\frac{A'(r_1(t))B(r_1(t))-\frac{1}{2}}{A'(r_1(t))\widetilde{B}(r_1(t))}=\frac{\widetilde{A}(z(t))}{A(z(t))}
\end{aligned}$$

Differentiating (4.36) with respect to t we get

$$\begin{aligned}
LHS=r_1'(t)\frac{\frac{1}{2}\Big(A''(r_1(t))\widetilde{B}(r_1(t))+A'(r_1(t))\widetilde{B}'(r_1(t))\Big)}{A'(r_1(t))^2\widetilde{B}(r_1(t))^2}+\\
r_1'(t)\frac{A'(r_1(t))^2\Big(B'(r_1(t))\widetilde{B}(r_1(t))-B(r_1(t))\widetilde{B}'(r_1(t))\Big)}{A'(r_1(t))^2\widetilde{B}(r_1(t))^2} \qquad (4.37)
\end{aligned}$$

Since

$$A''(r_1(t))=-\frac{N-1}{r_1(t)}A'(r_1(t))+a(r_1(t))A(r_1(t)). \qquad (4.38)$$

Moreover from $G_{rr}(r,s)+\frac{N-1}{r}G_r(r,s)+a(r)G(r,s)=0$ for $r<s$ we get

$$\begin{aligned}
&B'(r_1(t))\widetilde{B}(r_1(t))-B(r_1(t))\widetilde{B}'(r_1(t))= \qquad (4.39)\\
&\frac{A(r_1(t))B'(r_1(t))\widetilde{A}(r_1(t))\widetilde{B}(r_1(t))-A(r_1(t))B(r_1(t))\widetilde{A}(r_1(t))\widetilde{B}'(r_1(t))}{A(r_1(t))\widetilde{A}(r_1(t))}=\\
&\Big(\text{ and since }\widetilde{A}(r_1(t))\widetilde{B}(r_1(t))=A(r_1(t))B(r_1(t))\Big)=\\
&\frac{\Big(A(r_1(t))B'(r_1(t))-\widetilde{A}(r_1(t))\widetilde{B}'(r_1(t))\Big)B(r_1(t))}{\widetilde{A}(r_1(t))}=\\
&(\text{ by (2.17)})=-\frac{B(r_1(t))}{\widetilde{A}(r_1(t))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&LHS=\\
&r_1'(t)\frac{\frac{1}{2}\Big[A'(r_1(t))\Big(\widetilde{B}'(r_1(t))-\frac{N-1}{r_1(t)}B(r_1(t)\Big)+a(r_1(t))A(r_1(t))\widetilde{B}(r_1(t))\Big]-A'(r_1(t))^2\frac{B(r_1(t))}{\widetilde{A}(r_1(t))}}{A'(r_1(t))^2\widetilde{B}(r_1(t))^2}=\\
&(\text{ using (2.19)})= \qquad (4.40)\\
&r_1'(t)\frac{a(r_1(t))A(r_1(t))\widetilde{B}(r_1(t))-A'(r_1(t))^2\frac{B(r_1(t))}{\widetilde{A}(r_1(t))}}{2A'(r_1(t))^2\widetilde{B}(r_1(t))^2}=r_1'(t)\frac{a(r_1(t))A(r_1(t))^2-A'(r_1(t))^2}{2A'(r_1(t))^2A(r_1(t))\widetilde{B}(r_1(t))}
\end{aligned}$$

On the other hand, differentiating the RHS of (4.36) we get

$$RHS = \frac{\widetilde{A}'(z(t))A(z(t)) - \widetilde{A}(z(t))A'(z(t))}{A(z(t))^2} = \tag{4.41}$$
$$\frac{\widetilde{A}'(z(t))A(z(t))B(z(t))\widetilde{B}(z(t)) - \widetilde{A}(z(t))A'(z(t))B(z(t))\widetilde{B}(z(t))}{A(z(t))^2 B(z(t))\widetilde{B}(z(t))} =$$
$$\frac{\widetilde{A}'(z(t))\widetilde{B}(z(t)) - A'(z(t))B(z(t))}{A(z(t))\widetilde{B}(z(t))} = -\frac{1}{A(z(t))\widetilde{B}(z(t))}$$

From (4.40) and (4.41) we derive

$$r_1'(t) = -\frac{2A'(r_1(t))^2 A(r_1(t))\widetilde{B}(r_1(t))}{\left(a(r_1(t))A(r_1(t))^2 - A'(r_1(t))^2\right) A(z(t))\widetilde{B}(z(t))}. \tag{4.42}$$

which proves (4.34). In the same way we have (4.35).

■

Lemma 4.6. *Let us assume that r_1, r_2 satisfy (4.1), (4.2), (4.7) and (4.33). Moreover we assume that (1.11). Then we have that*

$$\frac{d}{dt}\left(u_{1,t}'(z(t)) + u_{2,t}'(z(t))\right)\Big|_{t=0} \neq 0. \tag{4.43}$$

Proof. We have that

$$u_{1,t}(r) \to \frac{G(r, r_1(t)) - \frac{G(z(t),r_1(t))}{G(z(t),r_2(t))}G(z(t), r_2(t))}{H(r_1(t), r_1(t)) - \frac{G(z(t),r_1(t))}{G(z(t),r_2(t))}G(r_1(t), r_2(t))} \tag{4.44}$$

Hence

$$u_{1,t}'(z(t)) \to \frac{\widetilde{A}'(z(t))\widetilde{B}(r_1(t)) - \frac{\widetilde{A}(z(t))\widetilde{B}((r_1(t))}{A(z(t))}A'(z(t))}{\frac{\widetilde{A}(r_1(t)A(z(t)) - \widetilde{A}(z(t)A(r_1(t))}{A(z(t)}\widetilde{B}(r_1(t))} =$$
$$\frac{\widetilde{A}'(z(t))A(z(t)) - \widetilde{A}(z(t))A'(z(t))}{\widetilde{A}(r_1(t)A(z(t)) - \widetilde{A}(z(t)A(r_1(t))} = \frac{\widetilde{A}'(z(t))\widetilde{B}(z(t)) - A'(z(t))B(z(t))}{\widetilde{A}(r_1(t)\widetilde{B}(z(t)) - A(r_1(t))B(z(t)} =$$
$$\text{(using (2.17))} = -\frac{1}{\widetilde{A}(r_1(t)\widetilde{B}(z(t)) - A(r_1(t))B(z(t))}. \tag{4.45}$$

In the same way we have that

$$u_{2,t}(r) \to \frac{G(r, r_2(t)) - \frac{G(z(t),r_2(t))}{G(z(t),r_1(t))}G(z(t), r_1(t))}{H(r_2(t), r_2(t)) - \frac{G(z(t),r_2(t))}{G(z(t),r_1(t))}G(r_2(t), r_1(t))}, \tag{4.46}$$

and then

$$u'_{2,t}(z(t)) \to \frac{A'(z(t))B(r_2(t)) - \frac{A(z(t))B((r_2(t))}{\widetilde{A}(z(t))}\widetilde{A}'(z(t))}{\frac{A(r_2(t)\widetilde{A}(z(t)) - A(z(t)\widetilde{A}(r_2(t))}{\widetilde{A}(z(t))}B(r_2(t))} =$$

$$-\frac{A'(z(t))\widetilde{A}(z(t)) - A(z(t))\widetilde{A}'(z(t))}{A(r_2(t)\widetilde{A}(z(t)) - A(z(t)\widetilde{A}(r_2(t))} = \tag{4.47}$$

$$-\frac{1}{A(r_2(t)B(z(t)) - \widetilde{B}(z(t))\widetilde{A}(r_2(t))}$$

Note that by (4.22), (4.45), (4.47)

$$u'_{1,0}(z)) = u'_{2,0}(z)) \Leftrightarrow \tag{4.48}$$

$$A(r_2(t)B(z(t)) - \widetilde{B}(z(t)\widetilde{A}(r_2(t)) = \widetilde{A}(r_1(t)\widetilde{B}(z(t)) - A(r_1(t))B(z(t))$$

Let us compute $\frac{d}{dt}\left(u'_{1,t}(z(t)) - u'_{2,t}(z(t))\right)\Big|_{t=0}$.

From (4.45), (4.47), (4.48) we get

$$\frac{d}{dt}\left(u'_{1,t}(z(t)) + u'_{2,t}(z(t))\right) = \tag{4.49}$$

$$-\frac{d}{dt}\left(\frac{1}{\widetilde{A}(r_1(t)\widetilde{B}(z(t)) - A(r_1(t))B(z(t))} - \frac{1}{A(r_2(t)B(z(t)) - \widetilde{B}(z(t))\widetilde{A}(r_2(t))}\right) =$$

$$\frac{1}{\left(\widetilde{A}(r_1(t)\widetilde{B}(z(t)) - A(r_1(t))B(z(t))\right)^2}\Big[\left(\widetilde{A}'(r_1(t)\widetilde{B}(z(t)) - A'(r_1(t))B(z(t))\right)r'_1(t) -$$

$$\left(A'(r_2(t)B(z(t)) - \widetilde{A}'(r_2(t))\widetilde{B}(z(t))\right)r'_2(t) + \left(\widetilde{A}(r_1(t) + \widetilde{A}(r_2(t)\right)\widetilde{B}'(z(t) -$$

$$(A(r_1(t) + A(r_2(t))\,B'(z(t)\Big]. \tag{4.50}$$

Note that by (4.36) we get

$$\widetilde{A}'(r_1(t))\widetilde{B}(r_1(t)) - \frac{\widetilde{A}(z(t)\widetilde{B}(r_1(t))}{A(z(t))}A'(r_1(t)) = -\frac{1}{2} \Leftrightarrow \tag{4.51}$$

$$\widetilde{A}'(r_1(t))A(z(t)) - A'(r_1(t))\widetilde{A}(z(t)) = -\frac{A(z(t))}{2\widetilde{B}(r_1(t))} \Leftrightarrow$$

$$\widetilde{A}'(r_1(t))\widetilde{B}(z(t)) - A'(r_1(t))B(z(t)) = -\frac{\widetilde{B}(z(t))}{2\widetilde{B}(r_1(t))}$$

In the same way we have that

$$A'(r_2(t)B(z(t)) - \widetilde{A}'(r_2(t))\widetilde{B}(z(t)) = \frac{B(z(t))}{2B(r_2(t))} \tag{4.52}$$

Finally by (4.48) we have

$$\left(\widetilde{A}(r_1(t)+\widetilde{A}(r_2(t)\right)\widetilde{B}'(z(t))-(A(r_1(t)+A(r_2(t))\,B'(z(t))=$$
$$(A(r_1(t))+A(r_2(t)))\left(\widetilde{B}'(z(t)\frac{B(z(t)}{\widetilde{B}(z(t)}-B'(z(t))\right)=$$
$$\frac{A(r_1(t))+A(r_2(t))}{A(z(t))} \tag{4.53}$$

Then (4.49) becomes

$$\frac{d}{dt}\left(u'_{1,t}(z(t))+u'_{2,t}(z(t))\right)=\frac{1}{\left(\widetilde{A}(r_1(t)\widetilde{B}(z(t))-A(r_1(t))B(z(t))\right)^2}\cdot$$
$$\left[-\frac{\widetilde{B}(z(t))}{2\widetilde{B}(r_1(t))}r'_1(t)-\frac{B(z(t))}{2B(r_2(t))}r'_2(t)+\frac{A(r_1(t))+A(r_2(t))}{A(z(t))}\right] \tag{4.54}$$

Using (4.34) and (4.36) we get

$$\left(\widetilde{A}(r_1)\widetilde{B}(z)-A(r_1)B(z)\right)^2\frac{d}{dt}\left(u'_{1,t}(z(t))+u'_{2,t}(z(t))\right)\Big|_{t=0}=$$
$$-\frac{\widetilde{B}(z)}{2\widetilde{B}(r_1)}r'_1(0)-\frac{B(z)}{2B(r_2)}r'_2(0)+\frac{A(r_1)+A(r_2)}{A(z)}= \tag{4.55}$$
$$-\frac{A'(r_1)^2A(r_1)}{\left(A'(r_1)^2-a(r_1)A(r_1)^2\right)A(z)}-\frac{\widetilde{A}'(r_2)^2\widetilde{A}(r_2)}{\left(\widetilde{A}'(r_2)^2-a(r_2)\widetilde{A}(r_2)^2\right)\widetilde{A}(z)}+\frac{A(r_1)+A(r_2)}{A(z)}=$$
$$\left(\text{from Lemma 4.1 and recalling that by (1.12) and (1.13) we get }\alpha_1=\frac{A'(r_1)}{A(r_1)}\right)=$$
$$-\frac{\frac{\alpha_1^2}{4}A(r_1)B(r_2)}{\left(\frac{\alpha_1^2}{4}-a(r_1)\right)A(z)B(r_2)}-\frac{\frac{\alpha_2^2}{4}\widetilde{A}(r_2)\widetilde{B}(r_1)}{\left(\frac{\alpha_2^2}{4}-a(r_2)\right)\widetilde{A}(z)\widetilde{B}(r_1)}+\frac{A(r_1)B(r_2)+A(r_2)B(r_2)}{A(z)B(r_2)}=$$
$$-\frac{\frac{\alpha_1^2}{4}G(r_1,r_2)}{\left(\frac{\alpha_1^2}{4}-a(r_1)\right)G(z,r_2)}-\frac{\frac{\alpha_2^2}{4}G(r_2,r_1)}{\left(\frac{\alpha_2^2}{4}-a(r_2)\right)G(z,r_1)}+\frac{G(r_1,r_2)+H(r_2,r_2)}{G(z,r_2)}=$$
$$-\frac{G(r_1,r_2)}{G(z,r_2)}\left(\frac{\frac{\alpha_1^2}{4}}{\frac{\alpha_1^2}{4}-a(r_1)}+\frac{r_1^{N-1}}{r_2^{N-1}}\frac{\frac{\alpha_2^2}{4}}{\frac{\alpha_2^2}{4}-a(r_2)}\frac{\alpha_1}{\alpha_2}-1-\frac{H(r_2,r_2)}{G(r_1,r_2)}\right)=$$
$$-\frac{G(r_1,r_2)}{G(z,r_2)}\left(\frac{4a(r_1)r_2^{N-1}\alpha_2}{\alpha_1^2-4a(r_1)}+\frac{4a(r_2)r_1^{N-1}\alpha_1}{\alpha_2^2-4a(r_2)}+r_1^{N-1}\left(\frac{H(r_2,r_2)}{G(r_1,r_2)}\alpha_2-\alpha_1\right)\right).$$

Since $\widetilde{A}(r_1)\widetilde{B}(z)-A(r_1)B(z)\neq 0$ we have the claim.

Proof of Theorem 1.1

From the definition of the functions $u_{1,t,p}(r)$ and $u_{2,t,p}(r)$ in (4.29) and (4.30) and by Lemma 4.6 we get that there exists $\bar{t}\in(0,1)$ such that

$$u'_{1,\bar{t},p}(\bar{t})=-u'_{2,\bar{t},p}(\bar{t})$$