INFINITE LOOP SPACES

BY

J. F. ADAMS

Hermann Weyl Lectures
The Institute for Advanced Study

PRINCETON UNIVERSITY PRESS

AND

UNIVERSITY OF TOKYO PRESS

PRINCETON, NEW JERSEY

1978

Copyright © 1978 by Princeton University Press

ALL RIGHTS RESERVED

Published in Japan exclusively by
University of Tokyo Press;
In other parts of the world by
Princeton University Press

Printed in the United States of America
by Princeton University Press, Princeton, New Jersey

Library of Congress Cataloging in Publication data will
be found on the last printed page of this book

HERMANN WEYL LECTURES

The Hermann Weyl Lectures are organized and sponsored by the School of Mathematics of the Institute for Advanced Study. Their aim is to provide broad surveys of various topics in mathematics, accessible to nonspecialists, to be published eventually in the Annals of Mathematics Studies.

The present monograph is the third in the series. It is an outgrowth of the sixth set of Hermann Weyl Lectures, which consisted of six lectures given by Professor J. F. Adams at the Institute for Advanced Study on March 18, 19, 24, 26, April 1, 2, 1975.

ARMAND BOREL

JOHN W. MILNOR

PREFACE

This book derives from a series of Hermann Weyl Lectures which I gave at the Institute for Advanced Study, Princeton, in the spring of 1975. It is a pleasure to thank my hosts for their invitation, their hospitality, and for providing so discriminating an audience. I should also apologize for my delay in submitting this manuscript. In the intervening time some progress has been made with the theory, and I have taken the opportunity to mention some of it below. Moreover, a number of other sources have appeared, and of these [96] and [99] can be recommended as particularly useful to experienced topologists who want to see the results of the subject. However, my object has been a more elementary exposition, which I hope may convey the basic ideas of the subject in a way as nearly painless as I can make it. In this the Princeton audience encouraged me; the more I found means to omit the technical details, the more they seemed to like it. If that is the reaction of seasoned topologists, I hope that beginners may find it useful to have a gentle introduction to the ideas used in the current literature.

I am very grateful to J. P. May, B. J. Sanderson and S. B. Priddy for reading the first draft of this book, in part or in whole; I have benefited greatly from their comments. It goes without saying that I accept the responsibility for any jokes which remain.

J. F. ADAMS

TABLE OF CONTENTS

Infinite Loop Spaces

CHAPTER 1
BACKGROUND AND PRELIMINARIES

§1.1. *Introduction*

In this introductory chapter I will begin with some historical remarks. At the same time I will sketch in some background material, familiar enough to the specialists, but necessary if the nonspecialist is to have a fair chance of reading as far as he wishes. I will sketch the existence of the following three fields.

(i) The study of infinite-loop-spaces.

(ii) The study of stable homotopy theory, via spectra.

(iii) The study of generalized homology and cohomology theories.

I will also try to explain the very close relations which hold between these three topics. Finally I will give a rough classification or survey of the spaces presently known to be infinite-loop-spaces.

§1.2. *Loop-spaces*

In this section I will introduce loop-spaces.

Let X be a space, with base-point x_0. By the loop-space ΩX, I mean the function-space

$$(X,x_0,x_0)^{(I,0,1)}$$

of continuous functions $\omega : I \to X$ from the unit interval $I = [0,1]$ to the space X which carry 0 to x_0 and 1 to x_0. We give it the compact-open topology; as its base-point, if it needs one, we take the function ω_0 constant at x_0. The functions $\omega \in \Omega X$ are called loops in X.

Two historical references are compulsory at this point. First we have the well-known work of Marston Morse. In [111] Morse considered a Riemannian manifold M, say connected; and he sought information about the set of geodesics in M from the point P to the point Q. He found a relationship between this set of geodesics and the topology of the space of all paths from P to Q; by the latter we mean the function space

$$(M,P,Q)^{(I,0,1)}$$

of continuous functions $\omega : I \to M$ which carry 0 to P and 1 to Q. The homotopy type of this space is actually independent of the choice of P and Q, so this space is equivalent to ΩM. For example, suppose M is a sphere S^n, with its usual Riemannian structure; then one knows how many geodesics there are from P to Q. (More precisely, one may go from P to Q by the shortest geodesic, of length say θ; but by starting in the same direction and failing to stop when one might, one may reach Q by a geodesic of length $2n\pi + \theta$; and by starting in the opposite direction, one gets geodesics of length $2n\pi - \theta$.) From this Morse was able to calculate the homology groups $H_*(\Omega S^n)$ of the loop-space ΩS^n. Running his method in the other direction, he then deduced that if you took the sphere S^n, and gave it some other Riemannian structure, you would still have an infinity of geodesics from P to Q.

Secondly we have the well-known work of Serre. In [129], Serre generalized the theorem which says that there are infinitely many geodesics from P to Q, so as to replace the sphere S^n by any complete Riemannian manifold whose homology is not that of a point. However this result is only one of the good things in this paper, and perhaps some of the others were even more important; I have in mind particularly the methods which Serre introduced.

To help in understanding the loop-space ΩX, Serre introduced the path-space

$$EX = (X,x_0)^{(I,0)} ,$$

that is, the space of continuous functions $f : I \to X$ which carry 0 to x_0. The space EX is contractible, but it is a useful intermediate between ΩX and X. Serre defined a continuous function

$$p : EX \to X$$

by

$$p(f) = f(1) ;$$

thus p assigns to each path its end-point. He showed that this function p has the homotopy lifting property for maps of cubes; as we now say, it is a fibering in the sense of Serre. The fibre $p^{-1}x_0$ is exactly the loop-space ΩX. We say that we have a Serre fibering

$$\Omega X \longrightarrow EX \xrightarrow{p} X .$$

(Here I interpose a note on notation. Out of historical piety I am following Serre, who used the letter E. Some later authors write PX for the path-space.)

This sort of construction has great potential generality; for by using it, Bourbaki later showed that any continuous map $f : X \to Y$ can be replaced by a fibering in the sense of Serre. To do so you have to replace X by another space which is homotopy-equivalent to X, but you can keep the same space Y.

Serre also showed that the apparatus of spectral sequences, due to Leray, carried over to this context. The use of spectral sequences to do homological calculations gave Serre's methods great technical power, and they were very successful; one may add that Serre's exposition was lucid and elegant, and so it is not surprising that his methods were widely copied.

Before I go on, I must recall some other background material which has been known for a long time. Perhaps the first thing which a homotopy-theorist knows about loop-spaces is that they allow him to manipulate homotopy groups by moving them from one dimension to the next. More precisely, we have

$$\pi_i(\Omega X) \cong \pi_{i+1}(X) .$$

We can proceed more generally. Let W be a further space, with base point w_0. Then maps

$$f : W \to X^I$$

are in (1-1) correspondence with maps

$$g : W \times I \longrightarrow X$$

in the following way:

$$(fw)(t) = g(w,t) \qquad (w \in W,\ t \in I) .$$

If we take account of the base-points, we find that maps

$$f : W, w_0 \longrightarrow \Omega X, \omega_0$$

are in (1-1) correspondence with maps

$$g : \Sigma W, \sigma_0 \longrightarrow X, x_0 .$$

Here ΣW is the quotient space obtained from $W \times I$ by identifying the subspace $(W \times 0) \cup (w_0 \times I) \cup (W \times 1)$ to a single point, which becomes the base-point σ_0 in ΣW. This quotient space is called the reduced suspension of W; it is often written SW.

Anyway, passing to homotopy classes we find a natural (1-1) correspondence

(1.2.1) $$[W, \Omega X] \longleftrightarrow [\Sigma W, X] .$$

Here I write $[U,V]$ for the set of homotopy classes of maps from U to V, where both maps and homotopies are supposed to preserve the base-point.

We would now express this by saying that the functors Σ and Ω are "adjoint." Of course this terminology is more recent, being due to Kan [75].

In particular we can take W to be the quotient space $I^n/\partial I^n$, where I^n is the unit cube in R^n and ∂I^n is its boundary. We get

$$\Sigma W = \frac{I^n \times I}{(I^n \times 0) \cup (\partial I^n \times I) \cup (I^n \times 1)} = \frac{I^{n+1}}{\partial I^{n+1}},$$

and our (1-1) correspondence becomes

$$\pi_n(\Omega X) \cong \pi_{n+1}(X) . \tag{1.2.2}$$

Of course, I have yet to explain why homotopy-theorists should want to move homotopy groups from one dimension to the next; this belongs to the next section.

§1.3. *Stable homotopy theory*

In this section I will introduce stable homotopy theory and spectra.

Topologists make a basic distinction between stable phenomena and unstable phenomena; a phenomenon is said to be stable if it can occur in any dimension, or any sufficiently large dimension, in a way which is essentially independent of the dimension. The construction which varies the dimension is usually suspension. For example, in homology-theory we have

$$\widetilde{H}_n(W;\pi) \cong \widetilde{H}_{n+1}(\Sigma W;\pi) ,$$

where $\widetilde{H}$ means reduced homology. But the principle is to be seen more clearly in homotopy-theory, where it goes back to Freudenthal [59]. For our purposes, let W and X be complexes of some sufficiently good sort, say CW-complexes. Then the suspension construction gives a function

$$\Sigma : [W,X] \longrightarrow [\Sigma W, \Sigma X] ;$$

and we have the following well-known theorem.

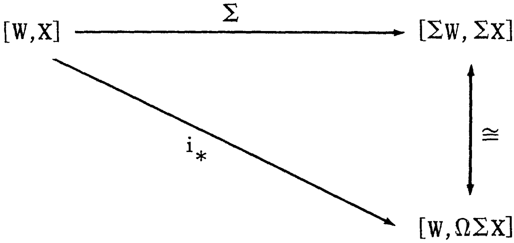
THEOREM 1.3.1. *Suppose that* X *is* (n–1)*-connected and* W *is of dimension* d; *then the function*

$$\Sigma : [W,X] \longrightarrow [\Sigma W, \Sigma X]$$

is onto if $d \leq 2n-1$, *and is a* (1–1) *correspondence if* $d \leq 2n-2$.

A suitable textbook reference is [135], especially p. 458.

To prove this theorem, we replace $[\Sigma W, \Sigma X]$ on the right by $[W, \Omega\Sigma X]$, using (1.2.1). We obtain the following commutative diagram.



Here i_* means the function induced by the map i, and $i : X \to \Omega\Sigma X$ is the map corresponding under (1.2.1) to the identity map $1 : \Sigma X \to \Sigma X$. This diagram allows one to prove the theorem by studying the loop-space $\Omega\Sigma X$ and its relation with X; similarly for other theorems in suspension-theory.

In fact, the study of loop-spaces proved a most valuable method in homotopy-theory. I. M. James [73] succeeded in replacing the rather large function-space ΩS^n by an explicit cell-complex which was so small that its structure could be well understood; I shall call this the "James model" for ΩS^n. It led him to good new results in suspension-theory. Bott and Samelson [44] succeeded in computing the homology of a number of loop-spaces, including the loop-space $\Omega(S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_d})$ on a wedge-sum or "bouquet" of spheres. Building on this, P. J. Hilton [63] succeeded in analyzing the homotopy type of $\Omega(S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_d})$; this again led to results in homotopy-theory.

Let us return to generalities. With the notation of (1.3.1), the homotopy classification problem of determining $[W,X]$ is said to be "stable" if $d \leq 2n-2$, so that we meet exactly the same problem for $[\Sigma W, \Sigma X]$, for $[\Sigma^2 W, \Sigma^2 X]$ and so on in higher dimensions. More generally, we may say that stable homotopy-theory is the part of homotopy-theory which studies phenomena which are stable in the intuitive sense described above. In order to persuade unbelievers, one has to show that it contains theorems of interest.

A topic which was studied quite early, and which provided a good advertisement for stable homotopy-theory, was Spanier-Whitehead duality [137, 134]. Suppose we have a good space X, say a finite complex, and we embed X in a sphere S^n in a way which is not pathological, so that the complement $\mathcal{C}X$ of X in S^n has a finite complex Y as a deformation retract, and similarly $\mathcal{C}Y$ has X as a deformation retract. Then the Alexander duality theorem tells us that the homology and cohomology of Y are determined by X and do not depend on the embedding of X in S^n. On the other hand the fundamental group $\pi_1(Y)$ is not determined by X and does depend on the embedding; it is sufficient to consider the example $X = S^1$, $n = 3$ (classical knots).

The question then arises, how much about Y is determined by X? The answer is that X determines the stable homotopy type of Y. Here we say that Y and Z are of the same stable homotopy type if there exists an integer m such that $\Sigma^m Y$ and $\Sigma^m Z$ are homotopy-equivalent. Alternatively, we can define a new category, the stable homotopy category of Spanier and Whitehead [136, 138], by saying that the objects shall be the finite complexes, and $\{Y,Z\}$, the set of stable homotopy classes of maps from Y to Z, shall be

$$\lim_{m\to\infty} [\Sigma^m Y, \Sigma^m Z] .$$

This limit is attained by (1.3.1); if Y were not finite-dimensional this definition of $\{Y,Z\}$ would not be appropriate. Then Y and Z are of the same stable homotopy type exactly when they are equivalent in the stable homotopy category.

The question about embeddings in S^n may now be answered more explicitly by saying that Y depends on X via a contravariant functor $D = D_n$. Here the functor $D = D_n$ takes values in the stable homotopy category of Spanier and Whitehead; it is defined on a full subcategory of that, for it is defined on all objects X which can be well embedded in S^n, and on all morphisms in the category between such objects X. Spanier and Whitehead call $D_n X$ the "n-dual" of X.

However, it was soon observed that the stable homotopy category constructed by Spanier and Whitehead does not contain sufficient objects for some purposes (even if we relax the assumption that our complexes are finite). A compelling example was provided by Thom's work on cobordism [151]. He reduced the study of cobordism groups to the study of certain stable homotopy groups

$$\lim_{n \to \infty} \pi_{n+r}(MO(n))$$

$$\lim_{n \to \infty} \pi_{n+r}(MSO(n))$$

and so on. Here the spaces $MO(n)$, $MSO(n)$ etc., are ones constructed by Thom, and usually called "Thom complexes"; the limits $\lim_{n \to \infty}$ can be formed because the spaces $MO(n)$, $MSO(n)$ and so on come provided with maps

$$\Sigma MO(n) \longrightarrow MO(n+1)$$

$$\Sigma MSO(n) \longrightarrow MSO(n+1)$$

and so on.

It was explicitly stated by John Milnor ([104], especially pp. 511-512) that one could argue very much more clearly if one could work in a category where there is a single object **MO** rather than the spaces MO(n) which approximate to it, and similarly a single object **MSO**, and so on. Now such a context was already known [81, 82]. For our purposes, a "spectrum" E is a sequence of spaces E_n (with base-point) provided with maps $\varepsilon_n : \Sigma E_n \to E_{n+1}$. For example, the spaces MO(n) provided with their maps $\Sigma MO(n) \to MO(n+1)$ constitute a spectrum, the Thom spectrum **MO**. Similarly for **MSO**. As another example, let X be a CW-complex with base-point; its "suspension spectrum" is the spectrum in which the n^{th} space is $\Sigma^n X$ and the maps are the identity maps $\Sigma(\Sigma^n X) = \Sigma^{n+1} X$. We shall write this spectrum $\Sigma^\infty X$; we would like to say that this defines a functor Σ^∞ from CW-complexes to spectra. Of course, for this purpose (and for other purposes) we have to make spectra into a category. One has to get the notion of a "map of spectra" right, in a way that is slightly technical. However, it is now generally agreed and understood that the best category in which to do stable homotopy theory is the category of spectra constructed by J. M. Boardman [34, 35, 36, 153], or else some category equivalent to Boardman's. I have tried to give an exposition in the most elementary terms in [9], especially pp. 131-146. For the moment I summarize the following points from it.

(i) A CW-spectrum is one in which each space E_n is a CW-complex (with base-point) and each map $\varepsilon : \Sigma E_n \to E_{n+1}$ embeds ΣE_n as a subcomplex of E_{n+1}. One can take the CW-spectra as the objects of the required category.

(ii) Σ^∞ is a functor; and whenever X is finite-dimensional it induces a (1-1)-correspondence

$$\lim_{n\to\infty} [\Sigma^n X, \Sigma^n Y] \longrightarrow [\Sigma^\infty X, \Sigma^\infty Y] .$$

(Here $[\Sigma^\infty X, \Sigma^\infty Y]$ means homotopy classes of maps in the category of CW-spectra.)

(iii) The category of CW-spectra is so arranged that one can carry over all the constructions which one usually performs with CW-complexes.

(iv) If you like you can make the category of CW-spectra into a graded category. Let E be a spectrum; then by reindexing it we can get a new spectrum; for example, we can define F by $F_n = E_{n+1}$. Then a map of degree 1 from E to G will be an ordinary map (of degree 0) from F to G. We write $[X,Y]_n$ for the set of homotopy classes of maps of degree n from X to Y.

This is sufficient to give the basic ideas. The important thing is to know that there is a good category of spectra, and not to insist on any one choice of details for its construction. In fact there are alternative ways of setting up the details; they all lead to the good category, but for some particular application one may have an advantage over another. Let us keep our options open.

This completes the introduction of stable homotopy theory and spectra; we are now ready to go back and talk about loop-spaces again.

§1.4. *Infinite loop spaces*

In this section I will introduce infinite loop spaces.

A loop-space is better than an ordinary space; not every space is equivalent to a loop-space. For example, a loop-space is an H-space, and not every space is an H-space. Here I recall that a space X is an H-space if it comes equipped with a product map

$$\mu : X \times X \longrightarrow X$$

which satisfies appropriate axioms. Equivalently, we may suppose given for each W a product operation on the set [W,X], so that this product is natural for maps of W. The minimum axiom is that the map from W to X which is constant at the base-point should act as a unit for this product; equivalently, the base-point in X should act as a unit (up to homotopy) for the map μ. (All maps and homotopies are supposed to preserve the base-point.)

It is clear that a loop-space $X = \Omega Y$ is an H-space. In fact, we can