PROJECTIVE GEOMETRY

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of n dimensions

VOLUME TWO OF INTRODUCTION TO MODERN ALGEBRA AND MATRIX THEORY

BY

OTTO SCHREIER

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FIRST ENGLISH-LANGUAGE EDITION

THE PRESENT WORK AND INTRODUCTION TO MODERN ALGEBRA AND MATRIX THEORY TOGETHER CONSTITUTE A COMPLETE AND UNABRIDGED TRANSLATION OF THE GERMAN-LANGUAGE BOOK EINFUEHRUNG IN DIE ANALYTISCHE GEOMETRIE UND ALGEBRA (VOLUMES ONE AND TWO) BY OTTO SCHREIER AND EMANUEL SPERNER. THE PRESENT WORK IS TRANSLATED BY THE LATE PROFESSOR CALVIN A. ROGERS.

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EDITOR'S PREFACE

With the present publication of *Projective Geometry,* the project of translating the famous German-language textbook *Einfuhrung in die analytische Geometrie und Algebra,* by Otto Schreier and Emanuel Sperner, originally published in two volumes, is now complete. As is well known, the purpose of that textbook was to offer a course in Algebra and Analytic Geometry which, when supplemented with a course on the Calculus, would give the student all he needs for a profitable continuation of his studies in modern mathematics. The Preface to the German Edition (see below) gives a more detailed description of the two volumes.

The only change that has been made has been to divide the two volumes somewhat differently in order that they might be usable independently. The first volume and the early part of the second volume were combined into a single book under the title *Introduction to Modern Algebra and Matrix Theory.* The balance, consisting of the major portion of the second volume is published herewith as *Projective Geometry of n Dimensions.* The titles of the two books indicate their respective contents.

The chief prerequisite for reading the present book, aside from a few elementary facts about affine space and systems of linear equations, is a knowledge of the elements of matrix theory such as is contained, for example, in the first four sections of Chapter V (Linear Transformations and Matrices) of *Introduction to Modern Algebra and Matrix Theory.*

Professor Calvin A. Rogers, the translator of the present volume, died before the preparation of the manuscript for the press was begun. The numerous questions that always call for consultation between editor and translator were referred to Professor Abe Shenitzer, whom the Editor wishes to thank for his very considerable help. The Editor also wishes to thank Professor F. Steinhardt. The final form of the manuscript is, of course, the responsibility of the Editor alone.

FROM THE PREFACE TO THE GERMAN EDITION

Otto Schreier had planned, a few years ago, to have his lectures on Analytic Geometry and Algebra published in book form. Death overtook him in Hamburg on June 2, 1929, before he had really begun to carry out his plan. The task of doing this fell on me, his pupil. I had at my disposal some sets of lecture notes taken at Schreier's courses, as well as a detailed (if not quite complete) syllabus of his course drawn up at one time by Otto Schreier himself. Since then, I have also given the course myself, in Hamburg, gaining experience in the process.

In writing this textbook,[1] which is to be published in two volumes, I have followed Schreier's own presentation as closely as possible, so that it might retain the characteristics impressed on the subject matter in Otto Schreier's treatment. In particular, as regards choice and arrangement of material, I have followed Schreier's outline faithfully, except for a few changes of minor importance.

This textbook is motivated by the idea of offering the student, in two basic courses on Calculus and Analytic Geometry, all that he needs for a profitable continuation of his studies in accordance with modern requirements. It is evident that this implies a stronger emphasis than has been customary on algebra, in line with the recent developments in that subject.

The prerequisites for reading this book are few indeed. For the early parts, a knowledge of the real number system—such as is acquired in the first few lectures of almost any calculus course—is sufficient. The later chapters make use of some few theorems on continuity of real functions and on sequences of real numbers. These also will be familiar to the student from the calculus. In some sections which give intuitive interpretations of the subject matter, use is made of some well-known theorems of elementary geometry, whose derivation on an axiomatic basis would of course be beyond the scope of this text.

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[1] See the preceding Editor's Preface.

What the book contains may be seen in outline by a glance at the table of contents. The student is urged not to neglect the exercises at the end of each section; among them will be found many an important addition to the material presented in the text.

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The authors' earlier book on matrices has been incorporated into [Chapter V of *Introduction to Modern Algebra and Matrix Theory*],[1] with a few re-arrangements and omissions in order to achieve a more organic whole. The arrangement of material in this chapter is such that the first four sections of the chapter contain essentially all that is needed for [*Projective Geometry*].

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To Mr. W. Blaschke (Hamburg) I owe a debt of gratitude for his continuous interest and help. I also wish to thank Messrs. O. Haupt (Erlangen) and K. Henke (Hamburg) for many valuable hints and suggestions. In preparing the manuscript, I have had the untiring assistance of my wife. For reading the proofs I am indebted to Mr. H. Bückner (Königsberg) in addition to those named above.

Königsberg, October 1935

EMANUEL SPERNER

[1] See the preceding Editor's Preface.

CONTENTS

CHAPTER I

n-DIMENSIONAL PROJECTIVE SPACE

For certain geometrical questions, whose study is central to this book, it is advantageous to extend affine (or euclidean) space by adding to it certain new points, the so-called points at infinity. This procedure is suggested by quite elementary geometrical facts. For example, in order to avoid the oftentimes awkward distinction between intersecting and parallel lines in a plane, we are tempted to ascribe to parallel lines a point of intersection 'at infinity.' Another case in point is afforded by the fact when one line of the affine plane is projected onto another by means of central projection[1] this does not in general establish a one-to-one correspondence between the points of these two lines, whereas it may be made into such a correspondence by an appropriate adjunction of points at infinity. The same is true for the central projection of two planes in space upon each other.

Our immediate task, then, will be to establish and to give a precise analytic description of the introduction of these points at infinity.

Extension of the Affine Plane to the Projective Plane

Because of its intuitive appeal, we shall start with the two-dimensional case.

We shall first of all introduce new coordinates in the affine plane (the so-called homogeneous coordinates). In doing this, we begin with

[1] The central projection upon each other of two lines g and h with respect to a center of projection S is defined by the following rule: P, on g, is taken as the image of Q, on h, and conversely, Q is taken as the image of P, if P, Q, and S lie on a line.

It is therefore clear that P_0 on g has no image point on h if P_0S is parallel to h. Similarly, Q_0 has no image point on g if Q_0S is parallel to g. If we let P_0 correspond to *one* point at infinity on h and Q_0 to *one* point at infinity on g, then exactly one point of h is associated with each point of g, and conversely linear coordinates and hence take as our starting point a *fixed* linear coordinate system in the plane. We get in this way a definite one-to-one correspondence between the points of the plane and the ordered pairs of real numbers. If a point P has the coordinates x_1, x_2, we write $P = (x_1, x_2)$.

Next, we consider all the *ordered triples* of real numbers (ξ_0, ξ_1, ξ_2) for which $\xi_0 \neq 0$. These number triples and the points of the plane are now put into correspondence by means of the following rule:

$P = (x_1, x_2)$ and an ordered triple (ξ_0, ξ_1, ξ_2) with $\xi_0 \neq 0$ are to correspond to each other if and only if:

$$x_1 = \frac{\xi_1}{\xi_0}, \quad x_2 = \frac{\xi_2}{\xi_0}.$$

It follows immediately from this that to each triple (ξ_0, ξ_1, ξ_2) there corresponds only one point, namely, the point with linear coordinates $\frac{\xi_1}{\xi_0}, \frac{\xi_2}{\xi_0}$. On the other hand, to each point $P = (x_1, x_2)$ there correspond infinitely many number-triples. For, the point P obviously corresponds to the triples (ξ_0, ξ_1, ξ_2) and $(\lambda\xi_0, \lambda\xi_1, \lambda\xi_2)$ for arbitrary real $\lambda \neq 0$, since

$$\frac{\xi_i}{\xi_0} = \frac{\lambda\,\xi_i}{\lambda\,\xi_0} \quad (i = 1, 2).$$

Furthermore, the following holds: If two number-triples (ξ_0, ξ_1, ξ_2) and (ξ_0', ξ_1', ξ_2') with $\xi_0, \xi_0' \neq 0$, correspond to the same point, then there exists a $\lambda \neq 0$ such that $\xi_i' = \lambda\,\xi_i,\ i = 0, 1, 2$. For from $\frac{\xi_i'}{\xi_0'} = \frac{\xi_i}{\xi_0}$ $(i = 1, 2)$ it follows immediately that $\xi_i' = \frac{\xi_0'}{\xi_0}\,\xi_i$. Thus, $\frac{\xi_0'}{\xi_0}$ is the desired λ.

Hence, it is also evident that all the triples corresponding to the same point may be obtained from a given one of them (ξ_0, ξ_1, ξ_2) by multiplying it by an arbitrary real $\lambda \neq 0$.

In particular, all the triples associated with the point $P = (x_1, x_2)$ are of the form $(\lambda, \lambda x_1, \lambda x_2)$, since $(1, x_1, x_2)$ is one particular triple of this kind.

Since the numbers ξ_i of one of our triples (ξ_0, ξ_1, ξ_2) uniquely determine the corresponding point, we may regard them as the coordinates of that point. The coordinates introduced with the help of this correspondence are called *homogeneous coordinates*, or *ratio coordinates* (since they are determined only up to a common constant of proportionality).