Chapter 1

Introduction

1.1 About the book

This book deals with certain robust control problems for a linear time invariant (LTI) infinite dimensional systems. Robust stabilization and sensitivity minimization problems (as well as disturbance attenuation in the sense of reducing the worst energy amplification from disturbance to an output signal) are studied in the framework of $\mathcal{H}^\infty$ control. In this setting the plant uncertainty is assumed to be dynamic (it has a transfer function). The book also includes a discussion on robust stabilization (stability margin optimization) under a parametric uncertainty. But the main focus is on the $\mathcal{H}^\infty$ control problem. An operator theoretic approach to this problem is presented here. This method, known as the *skew Toeplitz* theory, has been developed over the past few years (1987–1994), for several different cases: one, two and four block $\mathcal{H}^\infty$ optimal and suboptimal problems, stable and unstable plants, SISO (single input single output) and MIMO (multi input multi output) plants, etc. See the papers by H. Bercovici, C. Foias, A. Frazho, C. Gu, H. Özbay, M. C. Smith, A. Tannenbaum, O. Toker and G. Zames, [6], [28], [30], [33], [34], [35], [37], [38], [51], [79], [81], [82], [99], [101], etc. This book is based on these papers. The skew Toeplitz techniques have been applied to two benchmark problems: an unstable system with a time delay, and a flexible beam. These examples are described in our joint papers [20],

[67], [100] with D. Enns, K. Lenz, B. Morton, O. Toker and J. Turi. The flexible beam example is studied in Chapter 7. Several different time delay system examples appear in Chapters 4, 5, 6, and 7.

There are many other articles published on the $\mathcal{H}^\infty$ control of infinite dimensional systems. For example, the one block problem has been studied in [25], [48], [65], [64], [90], [114], [122]. Robust stabilization problem, for coprime factor perturbations of infinite dimensional systems, is a two block $\mathcal{H}^\infty$ problem and has been addressed in [42], [43], [71], [84], [112]. More general forms of $\mathcal{H}^\infty$ control problem for distributed parameter systems have been considered in [13], [27], [120], [92]. This list is not intended to be a complete literature survey on the $\mathcal{H}^\infty$ control of infinite dimensional systems. For a survey on this subject, see [14]. Most of the above mentioned papers approach the $\mathcal{H}^\infty$ control problem from the input/output operator theoretic point of view. The state space approach is more popular for the $\mathcal{H}^\infty$ control of finite dimensional systems. Because, in this case one can solve the problem from algebraic Riccati equations, which involve simple linear algebra, see e.g. [3], [17], [39], [47]. There are also game theoretic interpretations of these solutions, see e.g. [5] and references therein. Although it is possible to extend these results to infinite dimensional systems, in this case one has to be careful in using state space methods since more complicated semigroup theory and operator valued Riccati equations are involved. See [13] and [106] for the details of the state space $\mathcal{H}^\infty$ control problems for infinite dimensional systems. For a class of delay systems, numerical solutions to $\mathcal{H}^\infty$ control problems can also be obtained from finite dimensional Riccati equations, see e.g. [72] [95] and references therein.

In this book we shall see that, under certain mild assumptions, one can obtain $\mathcal{H}^\infty$ controllers (optimal and suboptimal), for distributed parameter systems, by solving a *set of finitely many linear equations*, which is called the *singular system*. Our purpose is to present basic steps of the skew Topelitz theory leading to these equations. We would like to emphesize that these finitely many equations are derived directly from the original infinite dimensional plant, i.e., no approximation is made. Recently H. Tu has developed a MATLAB program, [103], which constructs and solves these equations. Together with the formulae given in [80], this program computes the $\mathcal{H}^\infty$ optimal controllers for a class of distributed plants. For the same class of systems, O. Toker has obtained a much simplified version of the singular system equations, [98]–[101] to compute all suboptimal $\mathcal{H}^\infty$ controllers. A MATLAB based program can be obtained via e-mail by sending a request to H. Özbay.

The book is organized as follows. In Chapter 2 we give a mathematical background on linear operator theory and interpolation theory. Chapter 3 sets-up the $\mathcal{H}^\infty$ control problems related to robust stability and sensitivity minimization. Generalized stability margin optimization problem is also defined in this chapter. In Chapter 4, Nevanlinna-Pick interpolation approach to stability margin optimization, and optimal robustness/sensitivity problems, is presented. Also in this chapter, an operator theoretic approach is given for the solution of the standard one block $\mathcal{H}^\infty$ problem for stable distributed plants. In Chapter 5 we present generalizations of this solution to two block problem for unstable plants. Computation of suboptimal $\mathcal{H}^\infty$ controllers are discussed in Chapter 6. Two benchmark examples are given in Chapter 7. The status of the skew Toeplitz theory for the multivariable systems is discussed in Chapter 8, with the commutant lifting theorem. Finally in Chapter 9 we make some concluding remarks.

We have tried to keep the prerequisites to a minimum in writing this book to make it accessible to the widest possible control audience. We also tried to make the book accessible to mathematics audience, who may want to overlook certain explanatory paragraphs aimed at engineers. Basically what is needed is a good background in systems and some knowledge of $\mathcal{H}^\infty$ theory, say from [16]. We have tried to fill in most of the relevant mathematical details in order to make the book as self-contained as possible. However, courses in real and complex analysis will be very helpful to one's understanding. Some results presented in this book are stated without proofs for which the reader is referred to papers where they originally appear. The material of the book has already been course tested for second year control students at The Ohio State University and the University of Minnesota. The authors would like to thank Mr. Xing Guo, Dr. Thaddeus E. Peery and Dr. Onur Toker for carefully reading parts of the manuscript.

1.2 $\mathcal{H}^\infty$ control of distributed plants

The main reason to use feedback in the control of dynamical systems is to design against uncertainties. In a typical control system there are two kinds of uncertainties: modeling errors and disturbances. The purpose of feedback control is to achieve certain performance specifications in the closed loop system despite these uncertainties.

In control system design we start with a mathematical model of a given physical system. Infinite dimensional system models appear in many engineering applications where the physical system is spatially distributed, or contains time delays. For example, distillation columns [70], flexible beams [7], [68], heat conduction systems [105], aeroelastic systems [83] etc., can be cited as such engineering applications. For spatially distributed systems and systems with time delays partial differential equations or functional differential equations are taken as infinite dimensional mathematical models because these are the *simplest and most natural* representations of such systems, which give good physical insight. So, one of the reasons to use distributed models in the controller design is that infinite dimensional models may be more accurate in representing the dynamics of a physical system compared to finite dimensional models. On the other hand, in some cases infinite dimensional models which contain a few parameters are used for physical phenomenon which can otherwise be better explained by very high order finite dimensional models. Thus, the economical representation of the system is another important reason why distributed models are used in practice. For example there is only one parameter, h, in the representation of the time delay element e^{-hs}, which can be seen as an approximation to a finite dimensional model with many right half plane zeros, [20]. In general, transfer functions of distributed parameter systems are transcendental functions in the Laplace transform variable s, along with a few parameters, such as time delay, stiffness or damping coefficient of a beam.

In this book, the mathematical model describing a physical system will be assumed to be infinite dimensional. The controller has to be designed based on this *nominal plant model*. Since every model is an idealization of a much more complicated system, there is a modeling error. Of course it is impossible to characterize the error exactly (otherwise it would be possible to get an exact system description). On the other hand, it is possible to express modeling errors as perturbations of the nominal model, and most of the time it is possible to find an upper bound on these perturbations. Here we will consider perturbations of the nominal transfer function as modeling errors. Therefore, we are restricting ourselves to linear time invariant (LTI) perturbations of a LTI nominal model. Although this set-up ignores possible nonlinearities and time varying parameters in the actual system, it does handle an important class of modeling uncertainties. A nominal model transfer function $P(s)$, and a weighting function $W(s)$ (which represents an upper bound $|W(j\omega)|$ of the modeling error at each frequency $j\omega$), determine the "class of all possible plants," denoted by $\mathcal{P}$. We assume that the actual system, which is unknown, belongs to $\mathcal{P}$. Then, the robust stabilization problem is to find a fixed controller C, such that the closed loop system (represented by $[C, P_\Delta]$) is stable for all $P_\Delta \in \mathcal{P}$.

Besides stability, we should also study the effects of disturbances on the closed loop system behavior. We will assume that the disturbance is a finite energy signal. Then, the "effect" of the disturbance can be defined as the ratio of the output energy to the energy of the disturbance, i.e. energy amplification in the system. We can say that the closed loop system $[C, P]$ (resp. $[C, P_\Delta]$) has "good nominal (resp. robust) performance" if this energy amplification is "small" (resp. "small" for all $P_\Delta \in \mathcal{P}$).

In the text (see in particular Chapter 3), we will give definitions of robust stability, robust performance, and show that these problems can be put in the framework of the $\mathcal{H}^\infty$ control. This will allow us to formulate problems of robust system analysis and design in a *precise* mathematical manner which will make them amenable to techniques from operator theory and complex analysis. We will see that employing such a methodology, one can explicitly solve very general $\mathcal{H}^\infty$ problems in a simple, implementable manner. In fact, we will work our way up the hierarchy of control problems until we reach the general multivariable standard problem in Chapter 8.

Chapter 2

Mathematical Preliminaries

2.1 Notation

Below is the notation used throughout this book.

$\mathbf{Z}$: integers

$\mathbf{Z}_+$: non-negative integers, $\{n \in \mathbf{Z} \ : \ n \geq 0\} = \{0, 1, 2, \ldots\}$.

R : real numbers.

R_+ : non-negative real numbers, $\{t \in \mathsf{R} \ : \ t \geq 0\} = [0 \ , \ \infty)$.

$\mathbb{C}$: complex numbers.

$\mathbb{C}_+$: open right half plane in $\mathbb{C}$, $\{s \in \mathbb{C} \ : \ \text{Re } s > 0\}$.

$\overline{\mathbb{C}_+}$: closed right half plane, $\{s \in \mathbb{C} \ : \ \text{Re } s \geq 0\}$.

$\widetilde{\mathbb{C}_+}$: extended right half plane, $\overline{\mathbb{C}_+} \cup \{\infty\}$.

$j\mathsf{R}$: imaginary axis, $\{s \in \mathbb{C} \ : \ \text{Re } s = 0\}$.

$j\mathsf{R}_e$: extended imaginary axis: $\{j\omega \ : \ \omega \in \mathsf{R} \cup \{\infty\}\}$.

D : open unit disc, $\{z \in \mathbb{C} \ : \ |z| < 1\}$.

$\overline{\mathbb{D}}$: closed unit disc, $\{z \in \mathbb{C} : |z| \leq 1\}$.

$\mathbb{T}$: unit circle, $\{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

$\overline{\alpha}$: complex conjugate of $\alpha \in \mathbb{C}$.

ess sup : essential supremum with respect to Lebesgue measure.

A^T : transpose of the matrix A.

A^* : transpose of the complex conjugate of A; when $\mathbf{A}$ is an operator, $\mathbf{A}^*$ denotes the adjoint of $\mathbf{A}$.

$\overline{\sigma}(A)$: largest singular value of A.

$\sigma(\mathbf{A})$: spectrum of the operator $\mathbf{A}$.

$\|\mathbf{A}\|$: norm of $\mathbf{A}$.

$\sigma_e(\mathbf{A})$: essential spectrum of $\mathbf{A}$.

$\|\mathbf{A}\|_e$: essential norm of $\mathbf{A}$.

$\mathcal{L}^1(\mathsf{R}_+)$: Lebesgue space of integrable real functions on R_+.

$\mathcal{L}^2(\mathsf{R}_+)$: Lebesgue space of square integrable real functions on R_+.

$\mathcal{L}^\infty(\mathsf{R}_+)$: Lebesgue space of essentially bounded real functions on R_+.

ℓ^1 : Real valued absolutely summable sequences on $\mathbb{Z}_+$.

ℓ^2 : Real valued square summable sequences on $\mathbb{Z}$.

ℓ^2_+ : Real valued square summable sequences on $\mathbb{Z}_+$.

ℓ^∞ : Real valued bounded sequences on $\mathbb{Z}_+$.

$\mathcal{L}^\infty(j\mathsf{R})$: Lebesgue space of essentially bounded functions on $j\mathsf{R}$.

$\mathcal{H}^\infty(\mathbb{C}_+)$: Hardy space of $\mathcal{L}^\infty(j\mathsf{R})$ functions which admit bounded analytical extensions to $\mathbb{C}_+$.

$\mathcal{L}^2(j\mathsf{R})$: Lebesgue space of square integrable functions on $j\mathsf{R}$.

$\mathcal{H}^2(\mathbb{C}_+)$: Hardy space of $\mathcal{L}^2(j\mathbb{R})$ functions which admit analytical extensions to $\mathbb{C}_+$.

$\mathcal{H}^1(\mathbb{C}_+)$: Hardy space of absolutely integrable functions on $j\mathbb{R}$ which admit analytical extensions to $\mathbb{C}_+$.

$\mathcal{L}^\infty(\mathbb{T}), \mathcal{H}^\infty(\mathbb{D}), \mathcal{L}^2(\mathbb{T}), \mathcal{H}^2(\mathbb{D}), \mathcal{H}^1(\mathbb{D})$: replace $j\mathbb{R}$ with $\mathbb{T}$, and $\mathbb{C}_+$ with $\mathbb{D}$ in the above definitions of $\mathcal{L}^\infty(j\mathbb{R})$, $\mathcal{H}^\infty(\mathbb{C}_+)$, $\mathcal{L}^2(j\mathbb{R})$, $\mathcal{H}^2(\mathbb{C}_+)$, and $\mathcal{H}^1(\mathbb{C}_+)$, respectively.

$\mathcal{H}^\infty_{m\times n}(\mathbb{C}_+), \mathcal{H}^\infty_{m\times n}(\mathbb{D}), \mathcal{L}^\infty_{m\times n}(j\mathbb{R}), \mathcal{L}^\infty_{m\times n}(\mathbb{T})$: $m \times n$ matrix valued functions whose entries belong to $\mathcal{H}^\infty(\mathbb{C}_+)$, $\mathcal{H}^\infty(\mathbb{D})$, $\mathcal{L}^\infty(j\mathbb{R})$, $\mathcal{L}^\infty(\mathbb{T})$ respectively.

$\mathcal{H}^2_n(\mathbb{C}_+), \mathcal{H}^2_n(\mathbb{D}), \mathcal{L}^2_n(j\mathbb{R}), \mathcal{L}^2_n(\mathbb{T})$: $n \times 1$ vector valued functions with entries in $\mathcal{H}^2(\mathbb{C}_+)$, $\mathcal{H}^2(\mathbb{D})$, $\mathcal{L}^2(j\mathbb{R})$, $\mathcal{L}^2(\mathbb{T})$ respectively.

$\|G\|_p$: p-norm of G, when G is in $\mathcal{L}^p(j\mathbb{R})$, $\mathcal{H}^p(\mathbb{C}_+)$, $\mathcal{L}^p(\mathbb{T})$, $\mathcal{H}^p(\mathbb{D})$, etc.

$\mathcal{H}_1 \ominus \mathcal{H}_2$: orthogonal complement of $\mathcal{H}_2$ in $\mathcal{H}_1$, (where $\mathcal{H}_2$ is a subspace of a Hilbert space $\mathcal{H}_1$).

$\mathcal{K}^2_n$: $\mathcal{L}^2_n \ominus \mathcal{H}^2_n$ (could be on $\mathbb{T}$ or $j\mathbb{R}$).

$\mathbf{I}$: identity operator.

$\mathbf{P}_-$: orthogonal projection operator from $\mathcal{L}^2$ to $\mathcal{K}^2$

$\mathbf{P}_+ := \mathbf{I} - \mathbf{P}_-$.

$m\mathcal{H}^2(\mathbb{D}) := \{mf \ : \ f \in \mathcal{H}^2(\mathbb{D})\}$ where m is an inner function.

$\mathcal{H}(m) := \mathcal{H}^2(\mathbb{D}) \ominus m\mathcal{H}^2(\mathbb{D})$.

$\mathbf{P}_\mathcal{H}$: orthogonal projection onto a subspace $\mathcal{H}$ of $\mathcal{L}^2$ (on $\mathbb{T}$ or $j\mathbb{R}$).

2.2 Hardy spaces

A function $G(s)$, $s \in \mathbb{C}$, is in $\mathcal{H}^p(\mathbb{C}_+)$, $1 \le p \le \infty$, if

(i): G is analytic in $\mathbb{C}_+$,

(ii): it is defined almost everywhere on $j\mathbb{R}$, and

(iii): its p−norm defined by

$$\begin{aligned} \|G\|_p &= \sup_{\sigma>0} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} |G(\sigma + j\omega)|^p d\omega \right)^{1/p}, \quad (1 \le p < \infty) \\ &= (\text{ess} \sup_{\sigma>0,\ \omega\in\mathbb{R}} |G(\sigma + j\omega)|) \quad (p = \infty) \end{aligned}$$

is finite.

If G does not satisfy (i) but satisfies (ii) and (iii) with $\sigma = 0$, then it is in $\mathcal{L}^p(j\mathbb{R})$.

Similarly, a function $g(z)$, $z \in \mathbb{C}$, is in $\mathcal{H}^p(\mathbb{D})$, $1 \le p \le \infty$, if

(i): g is analytic in $\mathbb{D}$,

(ii): it is defined almost everywhere on $\mathbb{T}$, and

(iii): its p−norm defined by

$$\begin{aligned} \|g\|_p &= \sup_{r<1} \left(\frac{1}{2\pi} \int_{0}^{2\pi} |g(re^{j\theta})|^p d\theta \right)^{1/p}, \quad (1 \le p < \infty), \\ &= (\text{ess} \sup_{r<1,\ \theta\in[0,2\pi]} |g(re^{j\theta})|) \ (p = \infty) \end{aligned}$$

is finite.

If g does not satisfy (i) but satisfies (ii) and (iii) with $r = 1$, then it is in $\mathcal{L}^p(\mathbb{T})$.

The spaces $\mathcal{L}^2_n(j\mathbb{R})$ and $\mathcal{H}^2_n(\mathbb{C}_+)$ (resp. $\mathcal{L}^2_n(\mathbb{T})$ and $\mathcal{H}^2_n(\mathbb{D})$) $n \ge 1$, are Hilbert spaces, with the inner product

$$\begin{aligned} \langle G, F \rangle &:= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)^* G(j\omega) d\omega, \quad (\text{resp.} \\ \langle g, f \rangle &:= \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{j\theta})^* g(e^{j\theta}) d\theta \ ). \end{aligned}$$

Note that the Laplace transform of $\mathcal{L}^2(\mathbf{R}_+)$ is the Hardy space $\mathcal{H}^2(\mathbb{C}_+)$. One can also see $\mathcal{L}_n^2(\mathbf{T})$ as the discrete Fourier transforms of ℓ^2 sequences, e.g. $g \in \mathcal{L}_n^2(\mathbf{T})$ has an expansion

$$g(e^{j\theta}) = \sum_{k=-\infty}^{\infty} g_k e^{jk\theta},$$

with $g_k \in \mathbb{C}^n$ and

$$\|g\|_2^2 = \sum_{k=-\infty}^{\infty} g_k^T g_k < \infty.$$

The second Hardy space $\mathcal{H}_n^2(\mathbf{D})$, (a subspace of $\mathcal{L}_n^2(\mathbf{T})$), is the space of discrete Fourier transforms of ℓ_+^2 sequences, i.e. $g \in \mathcal{H}_n^2(\mathbf{D})$ if and only if $g \in \mathcal{L}_n^2(\mathbf{T})$ and $g_k = 0$ for $k < 0$; in this case the Z-transform $g(z) = \sum_{k=0}^{\infty} g_k z^k$ converges for all $z \in \mathbf{D}$.

It is also important to note that

$$\mathcal{H}^\infty(\mathbf{D}) = \mathcal{L}^\infty(\mathbf{T}) \cap \mathcal{H}^2(\mathbf{D}).$$

If $G \in \mathcal{L}_{m\times n}^\infty(j\mathbf{R})$ (resp. $g \in \mathcal{L}_{m\times n}^\infty(\mathbf{T})$) then its ∞-norm is defined as

$$\|G\|_\infty = \operatorname{ess}\sup_{\omega\in\mathbf{R}} \overline{\sigma}(G(j\omega)) \quad (\text{resp. } \|g\|_\infty = \operatorname{ess}\sup_{\theta\in[0,2\pi]} \overline{\sigma}(g(e^{j\theta}))).$$

The 2-norm of a vector valued function $G \in \mathcal{L}_n^2(j\mathbf{R})$ (resp. $g \in \mathcal{L}_n^2(\mathbf{T})$) is defined as

$$\begin{aligned} \|G\|_2 &= \left(\frac{1}{2\pi}\int_{-j\infty}^{+j\infty} G(j\omega)^* G(j\omega)\,d\omega\right)^{1/2}, \quad (\text{resp.} \\ \|g\|_2 &= \left(\frac{1}{2\pi}\int_0^{2\pi} g(e^{j\theta})^* g(e^{j\theta})\,d\theta\right)^{1/2} \quad). \end{aligned}$$

Any $n \times n$ matrix U whose entries are in $\mathcal{L}^\infty(j\mathbf{R})$ (or in $\mathcal{L}^\infty(\mathbf{T})$) with the property

$$\begin{aligned} U(j\omega)^* U(j\omega) = U(j\omega) U(j\omega)^* &= I_{n\times n} \text{ a.e. } \omega \in \mathbf{R} \text{ (or} \\ U(e^{j\theta})^* U(e^{j\theta}) = U(e^{j\theta}) U(e^{j\theta})^* &= I_{n\times n} \text{ a.e. } \theta \in [0, 2\pi] \text{)} \end{aligned}$$

is called *unitary.* Unitary matrices preserve the norm, i.e. if U is an $n \times n$ matrix valued function which is unitary then we have $\|UL\|_\infty = \|U^*L\|_\infty = \|L\|_\infty$ for all L in $\mathcal{L}^\infty_{n\times m}$ (of $j\mathsf{R}$ or T). For such U, we also have $\|UL\|_2 = \|U^*L\|_2 = \|L\|_2$, for all L in $\mathcal{L}^2_n$ (of $j\mathsf{R}$ or T).

In the rest of this book when we refer to Lebesgue or Hardy spaces on the imaginary axis, right half plane, unit circle, or unit disc we will drop the arguments ($j\mathsf{R}$), ($\mathbb{C}_+$), (T), and (D) whenever the meaning is clear from the context.

2.3 Conformal map between $\mathbb{C}_+$ and D

In this book, the systems are represented by their transfer functions, which are functions of the Laplace transform variable $s \in \mathbb{C}_+$ (in the case of continuous time systems) or functions of the Z-transform variable $z \in \mathsf{D}$ (for discrete time systems). Our solution to the $\mathcal{H}^\infty$ control problems will be derived using functions defined on the unit disc ($z-$plane). This does not limit us to discrete time systems, because we can transform a continuous time problem to a discrete time problem via a conformal map between $\mathbb{C}_+$ and D . A simple example of such a map is

$$z = \frac{s-a}{s+a}, \quad s = a\frac{1+z}{1-z}, \quad a > 0$$

where $s \in \mathbb{C}_+$ and $z \in \mathsf{D}$. This conformal map transforms every point in $\mathbb{C}_+$ to a unique point in D and vice versa, the imaginary axis (boundary of $\mathbb{C}_+$) is mapped to the unit circle (boundary of D). In particular, the points $j\infty$ and 0 in the $s-$plane are mapped to the points 1 and -1 in the $z-$plane.

Any function $F \in \mathcal{H}^\infty$ defined on $\mathbb{C}_+$ can be represented in terms of a function $f \in \mathcal{H}^\infty(\mathsf{D})$, and vice versa, e.g. choosing $a = 1$:

$$f(z) = F(\frac{1+z}{1-z}) \quad \text{and} \quad F(s) = f(\frac{s-1}{s+1}).$$

The conformal map between $\mathbb{C}_+$ and $\mathbb{D}$ preserves all the important properties of $F(s)$ as a bounded analytic function, e.g., $f(z)$ is a bounded analytic function on $\mathbb{D}$ and

$$\|F\|_\infty = \text{ess} \sup_{\omega \in \mathbb{R}} |F(j\omega)| = \text{ess} \sup_{\theta \in [0,2\pi]} |f(e^{j\theta})| = \|f\|_\infty .$$

In view of the above remarks, we can transform the problem data from $\mathbb{C}_+$ to $\mathbb{D}$. For example, choosing $a = 1$, if $P(s)$ represents the transfer function of the plant, then it can also be represented by $p(z) = P(\frac{1+z}{1-z})$, as a function defined on the unit disc. Conversely, if the controller is given as a function of z, i.e. $c(z)$, then, its transfer function can be obtained from the inverse map, i.e. $C(s) = c(\frac{s-1}{s+1})$.

2.4 Bounded linear operators

2.4.1 Operator norm and the essential norm

Consider two Banach spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, with norms denoted by $\|\cdot\|_{\mathcal{K}_1}$ and $\|\cdot\|_{\mathcal{K}_2}$, respectively. Let $\mathbf{L}$ be a linear operator from $\mathcal{K}_1$ to $\mathcal{K}_2$. Then, $\mathbf{L}$ is bounded if its norm, defined by

$$\|\mathbf{L}\| := \sup \left\{ \frac{\|\mathbf{L}x\|_{\mathcal{K}_2}}{\|x\|_{\mathcal{K}_1}} \quad : \quad x \in \mathcal{K}_1 \ , \quad x \neq 0 \right\},$$

is finite.

Suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ are two separable Hilbert spaces and $\mathbf{L}$ is a bounded linear operator from $\mathcal{K}_1$ to $\mathcal{K}_2$. Then the norm of $\mathbf{L}$ is given by

$$\|\mathbf{L}\| = \max\{\|\mathbf{L}\|_e \ , \ \sigma_{max}\},$$

where σ_{max} denotes the largest singular value of finite multiplicity, of $\mathbf{L}$, and $\|\mathbf{L}\|_e$ denotes the essential norm.

Recall that a singular value of an operator $\mathbf{L}$ is the positive square root of an eigenvalue of the operator $\mathbf{L}^*\mathbf{L}$; it is of finite multiplicity if the eigenvalue has finite multiplicity. To define the essential norm let $\langle\cdot,\cdot\rangle_1$, denote the inner product on $\mathcal{K}_1$. We say that a sequence $x_n \in \mathcal{K}_1$, $n = 1, 2, 3, \ldots$, *converges to zero weakly* if

$$\langle y, x_n\rangle_1 \to 0, \quad \text{as} \quad n \to \infty, \quad \text{for all} \quad y \in \mathcal{K}_1.$$

Then, the essential norm of $\mathbf{L}$ is given by

$$\|\mathbf{L}\|_e = \max\{\sqrt{\lambda} \quad : \quad \lambda \in \sigma_e(\mathbf{L}^*\mathbf{L})\},$$

where $\sigma_e(\mathbf{L}^*\mathbf{L})$ denotes the essential spectrum of $\mathbf{L}^*\mathbf{L}$ which consists of those $\lambda \in \mathbb{C}$, for which there exists a sequence $x_n \in \mathcal{K}_1$, with $\langle x_n \, , \, x_n\rangle_1 = 1$ for all $n = 1, 2, \ldots$, and $x_n \to 0$ weakly as $n \to \infty$, such that

$$\|(\lambda\mathbf{I} - \mathbf{L}^*\mathbf{L})x_n\|_{\mathcal{K}_1} \to 0 \quad \text{as} \quad n \to \infty.$$

2.4.2 $\mathcal{H}^\infty_{m\times n}$ as bounded linear operators on $\mathcal{H}^2_n$

We can see the elements of $\mathcal{H}^\infty_{m\times n}$ as bounded linear multiplication operators on $\mathcal{H}^2_n$. More precisely, if $G \in \mathcal{H}^\infty_{m\times n}$, then it defines a multiplication operator $\mathbf{M}_G : \mathcal{H}^2_n \to \mathcal{H}^2_m$

$$\mathbf{M}_G f = Gf \; , \quad f \in \mathcal{H}^2_n, \tag{2.1}$$

with the following property.

Theorem 1 *Let G be a matrix valued function in $\mathcal{H}^\infty_{m\times n}$. Then, $\mathbf{M}_G f \in \mathcal{H}^2_m$ for all $f \in \mathcal{H}^2_n$, and*

$$\|\mathbf{M}_G\| = \|G\|_\infty,$$

where $\mathbf{M}_G$ is the operator defined by (2.1).

Proof. We will give the proof for $n = m = 1$; but it can be extended to the general case easily. First note that multiplications of analytic functions give rise to analytic functions. Also, since both G and f are defined a.e. on the boundary (i.e. $j\mathbb{R}$ or $\mathbb{T}$), Gf is also defined a.e. on the boundary. Therefore, we just need to show that

$$\|\mathbf{M}_G\| = \sup\left\{\frac{\|Gf\|_2}{\|f\|_2} \quad : \quad f \in \mathcal{H}^2 \ , \ f \neq 0\right\} = \|G\|_\infty .$$

Note that for any $f \in \mathcal{H}^2$ we have

$$\begin{aligned}
\|Gf\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)f(j\omega)|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 |f(j\omega)|^2 d\omega \\
&\leq \|G\|_\infty^2 \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} |f(j\omega)|^2 d\omega\right) \\
&\leq \|G\|_\infty^2 \|f\|_2^2 .
\end{aligned}$$

Hence, $\|\mathbf{M}_G\| \leq \|G\|_\infty$. In order to see the converse recall the definition

$$\|G\|_\infty = \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} |G(j\omega)| .$$

This means that for every $\epsilon > 0$ there exists a finite number $\delta > 0$ and a measurable set Ω of measure δ, such that

$$|G(j\omega)| \geq \|G\|_\infty - \epsilon \quad \text{for all} \quad \omega \in \Omega \ .$$

On the other hand we can find a function $f_o \in \mathcal{H}^2$ such that $\|f_o\|_2 = 1$ and

$$|f_o(j\omega)| \geq (1-\epsilon)\sqrt{\frac{2\pi}{\delta}} \ \text{ for } \quad \omega \in \Omega \ .$$

One can construct such a function as follows. Let $f(s)$ in $\mathcal{H}^2(\mathbb{C}_+)$ be defined by the integral

$$f(s) = exp(\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{j\omega s - 1}{s - j\omega}\right)\frac{\varphi(\omega)d\omega}{\omega^2+1})$$

where

$$\varphi(\omega) = -\ln|f(j\omega)| \ .$$

So, given a desired magnitude function $\varphi(\omega)$ on Ω, such an $f_o(s)$ can be constructed. Thus, the 2-norm of $y_o = Gf_o$ is bounded below by

$$\begin{aligned} \|y_o\|_2 &\geq \left(\frac{1}{2\pi}\int_\Omega (\|G\|_\infty - \epsilon)^2 (\frac{2\pi}{\delta})(1-\epsilon)^2 d\omega\right)^{\frac{1}{2}} \\ &\geq (\|G\|_\infty - \epsilon)(1-\epsilon). \end{aligned}$$

Since ϵ can be made arbitrarily small we have

$$\|\mathbf{M}_G\| \geq \|G\|_\infty \ .$$

This concludes the proof for $n = m = 1$ □.

The above proof can be extended to multivariable case as follows: Again, it is easy to establish the upper bound. For the lower bound same argument works except that now for a fixed matrix $G(j\omega_o)$ we choose a fixed singular vector $f_o(j\omega_o)$ such that

$$\|G(j\omega_o)f_o(j\omega_o)\|^2 = \overline{\sigma}(G(j\omega_o))^2\|f_o(j\omega_o)\|^2.$$

For the case where the functions are defined on the unit disc $\mathbf{D}$, the proof is still valid with obvious modifications.

2.5 The shift operator

The *shift operator*, is defined as $\mathbf{S} : \mathcal{H}^2(\mathbf{D}) \to \mathcal{H}^2(\mathbf{D})$

$$(\mathbf{S}f)(z) = zf(z) = 0 + f_0 z^1 + f_1 z^2 + \ldots,$$

for all $f \in \mathcal{H}^2(\mathbf{D})$, with $f(z)$ being the Z-transform of the ℓ^2_+ sequence $\{f_k\}_{k=0}^{\infty}$. So, $\mathbf{S}$ "shifts the coefficients to the right."

The adjoint of the shift operator, denoted by $\mathbf{S}^*$, "shifts the coefficients to the left" as follows, $\mathbf{S}^* : \mathcal{H}^2(\mathsf{D}) \to \mathcal{H}^2(\mathsf{D})$

$$(\mathbf{S}^* f)(z) = z^{-1}(f(z) - f_0) = f_1 + f_2 z^1 + f_3 z^2 + \ldots$$

for all $f \in \mathcal{H}^2(\mathsf{D})$, as before.

An important point to remark is that $\mathbf{S}^{*k}\mathbf{S}^k$ is the identity (for any integer $k \geq 1$), however $\mathbf{S}^k\mathbf{S}^{*k} \neq \mathbf{I}$:

$$\begin{aligned}(\mathbf{S}^{*k}\mathbf{S}^k f)(z) &= f(z),\\ (\mathbf{S}^k\mathbf{S}^{*k} f)(z) &= f(z) - \sum_{\ell=0}^{k-1} f_\ell z^\ell = \sum_{\ell=k}^{\infty} f_\ell z^\ell.\end{aligned}$$

Note that $\mathbf{S}$ can be seen as the multiplication operator $\mathbf{M}_g$, where $g(z) := z$. Moreover, by Theorem 1 we have

$$\|\mathbf{S}\| = \|\mathbf{M}_g\| = \|g\|_\infty = \text{ess} \sup_{\theta \in [0,2\pi]} |e^{j\theta}| = 1.$$

Conversely, we can define multiplication operators using the shift operator. For example given any $g \in \mathcal{H}^\infty(\mathsf{D})$ the operator $g(\mathbf{S})$ is obtained by formally replacing z by $\mathbf{S}$ in the power series expansion of $g(z)$:

$$g(\mathbf{S}) = \sum_{k=0}^{\infty} g_k \mathbf{S}^k \quad \text{and} \quad (g(\mathbf{S})f)(z) = g(z)f(z)$$

for all $f \in \mathcal{H}^2(\mathsf{D})$. Note that by definition $g(\mathbf{S}) = \mathbf{M}_g$.

2.6 Inner-Outer factorizations

Definition: A function $m \in \mathcal{H}^\infty(\mathsf{D})$ is called *inner* if $|m(z)| \leq 1$ for all $z \in \mathsf{D}$ and $|m(e^{j\theta})| = 1$ a.e. $\theta \in [0, 2\pi]$.

Since inner functions have constant magnitude a.e. on T; the engineers will realize that they generalize *all pass* transfer functions. An important property of inner functions is that they do not change the norm, i.e. both $m \in \mathcal{H}^\infty(\mathbf{D})$ and $m^* \in \mathcal{L}^\infty(\mathbf{T})$ are unitary.

Examples of inner functions include

$$\begin{aligned} m_1(z) &= \frac{z-a}{1-az}, \quad a \in (-1,1) \\ m_2(z) &= e^{-h\frac{1+z}{1-z}}, \quad h>0 \\ m_3(z) &= \prod_{k=1}^{\infty}\left(\frac{z-a_k}{1-\overline{a_k}z}\right), \quad |a_k|<1, \quad \text{and} \quad \sum_{k=1}^{\infty}(1-|a_k|)<\infty \\ m_4(z) &= m_1(z)m_2(z)m_3(z). \end{aligned}$$

In the above examples we see that $m_2(z)$ and $m_4(z)$ are not defined at $z=1$. Also note that $a_\infty := \lim_{k\to\infty} a_k$ must lie on the unit circle, and at that point $m_3(z)$ and $m_4(z)$ are not well defined. Such points on the unit circle are the *essential singularities* of inner functions. We would like to point out that rational inner functions have no essential singularities.

Theorem 2 (Beurling) *Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}^2(\mathbf{D})$ which is invariant with respect to* $\mathbf{S}$ *(i.e. $\mathbf{S}\mathcal{M} = \{\mathbf{S}f \ : \ f \in \mathcal{M}\}$ is a subspace of $\mathcal{M}$). Then, there exists an inner function $m \in \mathcal{H}^\infty(\mathbf{D})$ such that*

$$m\mathcal{H}^2(\mathbf{D}) = \mathcal{M}.$$

Conversely, given any inner function m, the subspace $\mathcal{M}_m := m\mathcal{H}^2(\mathbf{D})$ is closed in $\mathcal{H}^2(\mathbf{D})$ and invariant under the shift operator $\mathbf{S}$. □

This theorem characterizes the shift invariant subspaces of $\mathcal{H}^2(\mathbf{D})$ in terms of the inner functions.

Definition: A function $g \in \mathcal{H}^\infty(\mathbf{D})$ is called *outer* if the closure of $g\ \mathcal{L}_+$ in $\mathcal{H}^2(\mathbf{D})$ is the whole space $\mathcal{H}^2(\mathbf{D})$, where $\mathcal{L}_+ = \{\sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{C}, \ \ n \geq 0\}$.

Outer functions generalize *minimum phase* functions: they don't have a zero in $\mathbf{D}$, but may have zeros on $\mathbf{T}$. So, if g is an outer function, with $\inf_\theta |g(e^{j\theta})| > 0$, then it is *invertible* in $\mathcal{H}^\infty(\mathbf{D})$, i.e. there is another outer function $h \in \mathcal{H}^\infty(\mathbf{D})$ such that $g(z)h(z) = 1$, for all $z \in \mathbf{D}$.

Theorem 3 *([94]) Let f be a function in $\mathcal{H}^\infty(\mathbf{D})$, then it admits an inner-outer factorization of the form*

$$f(z) = m(z)g(z),$$

where m is inner and g is outer. Note that $|f(e^{j\theta})| = |g(e^{j\theta})|$ a.e. $\theta \in [0, 2\pi]$ and hence $\|f\|_\infty = \|g\|_\infty$.□

An inner/outer factorization can be done by finding a spectral factor $g(e^{j\theta})$ of $|f(e^{j\theta})|^2$. Whenever $f(z)f(z^{-1})$ is a rational function, g is finite dimensional, and it can be found by solving an algebraic Riccati equation, see for example [39]. In the general case it might be difficult to find the inner/outer factorization. On the other hand, for several interesting situations where $f(z)f(z^{-1})$ is irrational, it is still possible to compute the inner/outer factorizations, see e.g. [67] for a flexible beam system example. Here we present a delay system example from [78].

Example: Let us consider

$$g(z) = \frac{e^{-h\frac{1+z}{1-z}}(\frac{1+z}{1-z} - 3)(\frac{1+z}{1-z} - 0.5)}{(\frac{1+z}{1-z} + 2)^2(\frac{1+z}{1-z} + 0.1 - e^{-h_1\frac{1+z}{1-z}})}, \quad h_1 = 2\ln(\frac{5}{3}), \quad h > 0.$$

Note that the only point in $\mathbf{D}$ where the term $(\frac{1+z}{1-z} + 0.1 - e^{-h_1\frac{1+z}{1-z}})$ becomes zero is $z = 1/3$. We can easily check that the multiplicity of this zero is 1. On the other hand the term $(\frac{1+z}{1-z} - 0.5)$, in the numerator, also becomes zero at $z = 1/3$. So, g is bounded in $\mathbf{D}$. The inner and outer parts of g are

$$\begin{aligned} m(z) &= e^{-h\frac{1+z}{1-z}}\left(\frac{z-0.5}{1-0.5z}\right) \\ f(z) &= g(z)/m(z). \end{aligned}$$

We conclude this section by noting that in control theory another classification of analytic functions is useful. A function $F(s)$ defined on $\mathbb{C}_+$ is called *proper* (resp. *strictly proper*) if

$$\lim_{|s|\to\infty} |F(s)| < \infty \quad (\text{resp.} \lim_{|s|\to\infty} |F(s)| = 0).$$

Similar definitions can be made for functions defined on $\mathbf{D}$, with respect to a distinguished point in $\mathbf{T}$.

2.7 The compressed shift operator

In order to define the compressed shift operator, we will first need a few elementary results about Hilbert spaces. First let $\mathcal{H}$ denote an arbitrary (complex, separable) Hilbert space, and $\mathcal{H}_1 \subset \mathcal{H}$ a Hilbert subspace. We define

$$\mathcal{H} \ominus \mathcal{H}_1 := \{h \in \mathcal{H} : \langle h, h_1 \rangle = 0 \quad \forall h_1 \in \mathcal{H}_1\}.$$

$\mathcal{H} \ominus \mathcal{H}_1$ is called the *orthogonal complement* of $\mathcal{H}_1$ in $\mathcal{H}$. One can show the following (see e.g. [54])

Theorem 4 *Let $h \in \mathcal{H}$. Then there exist unique vectors $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H} \ominus \mathcal{H}_1$, such that $h = h_1 + h_2$.*

Using the notation of Theorem 4, we define an operator $\mathbf{P}_1 : \mathcal{H} \to \mathcal{H} \ominus \mathcal{H}_1$ by setting $\mathbf{P}_1 h = h_2$ for each $h \in \mathcal{H}$. $\mathbf{P}_1$ is called the *orthogonal projection* of $\mathcal{H}$ onto $\mathcal{H} \ominus \mathcal{H}_1$.

Now given an inner function $m \in \mathcal{H}^\infty(\mathbf{D})$, by Theorem 2 (Beurling's theorem), $m\mathcal{H}^2(\mathbf{D}) \subset \mathcal{H}^2(\mathbf{D})$ is a closed shift-invariant subspace and every closed shift-invariant subspace of $\mathcal{H}^2(\mathbf{D})$ has this form, [94]. We can consider therefore the Hilbert space $\mathcal{H}(m) := \mathcal{H}^2(\mathbf{D}) \ominus m\mathcal{H}^2(\mathbf{D})$ and the corresponding orthogonal projection $\mathbf{P}_{\mathcal{H}(m)} : \mathcal{H}^2(\mathbf{D}) \to \mathcal{H}(m)$. So, any function $h \in \mathcal{H}^2(\mathbf{D})$ has an orthogonal decomposition $h = g + mf$ where $f \in \mathcal{H}^2(\mathbf{D})$ and $g \in \mathcal{H}(m)$. If $g \in \mathcal{H}(m)$ then m^*g is of the form

$$(m^*g)(\zeta) = \overline{m(\zeta)}g(\zeta) = \sum_{i=1}^{\infty} \phi_{-i}\zeta^{-i} \quad \text{for} \quad \zeta \in \mathbf{T} \tag{2.2}$$

where the right hand side converges a.e. on $\mathbf{T}$ and outside the unit disc, for some coefficients $\phi_{-i} \in \mathbf{C}, i \geq 1$ such that $\sum_{i=1}^{\infty} |\phi_{-i}|^2 < \infty$. In other words the function $g_{\perp} := m^*g$ is in $\mathcal{L}^2(\mathbf{T}) \ominus \mathcal{H}^2(\mathbf{D})$.

Before giving a precise definition of the compressed shift operator we would like to present some special properties of $\mathcal{H}(m)$ and $\mathbf{P}_{\mathcal{H}(m)}$ when m is rational. If m is a rational inner function then it is of the form $m = b_1 b_2$, where $b_1(z) = z^n$ and

$$b_2(z) = \prod_{k=1}^{\ell} \left(\frac{z - a_k}{1 - \overline{a_k} z} \right),$$

with $|a_k| < 1$, for $k = 1, \ldots, \ell$. When $m = b_1 b_2$, where b_1 and b_2 are as above, $\mathcal{H}(m) = \mathcal{H}(b_1 b_2)$ has an orthogonal decomposition of the form

$$\begin{aligned} \mathcal{H}(b_1 b_2) &= \mathcal{H}^2(\mathbf{D}) \ominus b_1 b_2 \mathcal{H}^2(\mathbf{D}) \\ &= (\mathcal{H}^2(\mathbf{D}) \ominus b_1 \mathcal{H}^2(\mathbf{D})) \oplus b_1(\mathcal{H}^2(\mathbf{D}) \ominus b_2 \mathcal{H}^2(\mathbf{D})) \\ &= \mathcal{H}(b_1) \oplus b_1 \mathcal{H}(b_2). \end{aligned}$$

Moreover, $\mathcal{H}(b_1)$ and $\mathcal{H}(b_2)$, (and hence $\mathcal{H}(b_1 b_2)$) are finite dimensional by the following results. For simplicity we will assume that $a_i \neq a_j$ for $i \neq j$, $1 \leq i, j \leq \ell$, i.e. the zeros of b_2 are distinct.

Lemma 1 *$\mathcal{H}(b_1)$ is a finite dimensional vector space of dimension n. A basis of $\mathcal{H}(b_1)$ consists of the elements $\{1,\ z, \ldots,\ z^{n-1}\}$.*

Proof. First note that the usual orthonormal basis for $\mathcal{H}^2(\mathbf{D})$ is given by $\{1,\ z,\ z^2, \ldots\}$. Then, in terms of these basis functions $b_1\mathcal{H}^2(\mathbf{D})$ has the basis $\{z^n,\ z^{(n+1)},\ z^{(n+2)}, \ldots\}$, because $b_1(z) = z^n$. Hence, $\{1,\ z, \ldots,\ z^{n-1}\}$ is an orthonormal basis for $\mathcal{H}(b_1)$. □